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5 **TORELLI ACTIONS AND SMOOTH STRUCTURES**  
 ON FOUR MANIFOLDS

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11 **ABSTRACT**

13 Artin presentations are discrete equivalents of planar open book decompositions  
 of closed, orientable three manifolds. Artin presentations characterize the fundamen-  
 15 tal groups of closed, orientable three manifolds. An Artin presentation also determines  
 a smooth, compact, simply connected four manifold that bounds the three dimensional  
 17 open book. In this way, the study of three and four manifolds may be approached purely  
 group theoretically. In the theory of Artin presentations, elements of the Torelli subgroup  
 19 act on the topology and smooth structures of the three and four manifolds. We show  
 that the Torelli action can preserve the continuous topological type of a four manifold  
 21 while changing its smooth structure. This is a new, group theoretic method of altering  
 the smooth structure on a four manifold.

*Keywords:* Four manifold; Artin presentation; smooth structure.

23 *Mathematics Subject Classification 2000:* 57R55, 57R57, 57M05

**1. Introduction**

25 In the theory of Artin presentations, a smooth, compact, simply connected  
 4-manifold, with a connected boundary, is already determined by an Artin pre-  
 27 sentation of the fundamental group of its boundary [20, 7]. Thus, the study of the  
 smooth structures of these 4-manifolds, a central problem of contemporary topology  
 29 and physics, and their Donaldson and Seiberg–Witten theories can be approached  
 in a new manner. This approach is systematic, exterior, purely group theoretic, and  
 31 transcends the Tietze-like methods of the Kirby calculus as well as the more or less  
 ad hoc interior surgery methods [18].

33 Elliptic complex surfaces, all of which have nontrivial Donaldson and Seiberg–  
 Witten invariants, can be presented by Artin presentations [7]. Therefore, Artin  
 35 presentation theory (AP theory) assures the metamathematical existence of a non-  
 trivial, discrete, purely group theoretic theory of Donaldson and Seiberg–Witten

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1 invariants. This is the beginning of a 4D analogue of such results already accom-  
 2 plished in 3D AP theory, starting with González-Acuña’s remarkable formula for  
 3 the Rohlin invariant [12, 7] and its generalization to the Casson invariant [5].

4 In AP theory, canonical subgroups of the Torelli groups (which are part of 2D  
 5 manifold theory) act on the topology and smooth structures of the 4D manifolds  
 6 above. This raises the important question: *can the Torelli action preserve the under-*  
 7 *lying topological structure but change the smooth structure of such a 4-manifold?*

8 The first aim of this paper is to show explicitly that this is possible. Discrete  
 9 Torelli group theory alone can “juggle” the smooth structures of a 4-manifold. In  
 10 other words, pure algebra (in fact, just discrete finitely presented group theory, with  
 11 its computer approachable methods) influences the mathematically and physically  
 12 important theory of smooth structures on simply connected 4-manifolds.

13 The second aim of this paper is to describe the general context into which these  
 14 examples fit and to present a pertinent conjecture. The canonical knot and link  
 15 theory of AP theory is conjectured to correspond strongly with smooth invariants  
 16 of the bounding 4-manifolds.

17 This paper is organized as follows. Sec. 2 contains an introduction to AP theory.  
 18 Section 3 constructs the Artin presentations described above, Sec. 4 identifies the  
 19 common boundary of these Artin presentations, and Sec. 5 discusses the knot theory  
 20 in these boundaries. Section 6 closes with open problems, a conjecture, and further  
 21 discussion of the Torelli action in AP theory.

## 2. AP Theory Background

23 An Artin presentation  $r$  is, by definition, a finite presentation:

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$$

24 satisfying the following equation in  $F_n$  (the free group on  $x_1, \dots, x_n$ ):

$$x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1) (r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n) \quad (\star)$$

25 Equation  $(\star)$  is deceptively simple, yet it ties together automorphisms of  $F_n$ ,  
 26 pure braids, closed orientable 3-manifolds, knots and links therein, and smooth com-  
 27 pact simply connected 4-manifolds. The reader is referred to [20, 7, 4] for detailed  
 28 proofs of statements made below.

29 Artin presentations arise naturally from homeomorphisms of  $\Omega_n$ , the compact  
 30 2-disk with  $n$  holes, that are the identity on the boundary. Any such homeomor-  
 31 phism defines a unique Artin presentation. The converse, which is more interesting,  
 32 is also true and was implicitly known to Artin himself in 1925 [3]. The main idea  
 33 runs as follows. Recall that  $P_n$ , the classical  $n$  strand pure braid group, has a faith-  
 34 ful representation as a group of automorphisms of  $F_n$ . The group of isotopy classes  
 35 (rel  $\partial$ ) of homeomorphisms of  $\Omega_n$  that are the identity on the boundary is canoni-  
 36 cally isomorphic to  $P_n \times \mathbb{Z}^n$  (framed pure braids). An Artin presentation  $r$  induces  
 37 an endomorphism of  $F_n$  by  $x_i \mapsto r_i^{-1} x_i r_i$ . Equation  $(\star)$  is precisely the condition  
 38

1 required to ensure this endomorphism of  $F_n$  is a pure braid automorphism. The  
 2 Artin presentation  $r$  also defines  $(a_i) \in \mathbb{Z}^n$  by  $a_i =$  exponent sum of  $x_i$  in  $r_i$ . In  
 3 this way,  $r$  determines  $h(r)$ , a unique (up to isotopy rel  $\partial$ ) self homeomorphism  
 4 of  $\Omega_n$  that is the identity on the boundary, which is the same as a framed pure  
 5 braid  $\beta(r)$ . Hence,  $\mathcal{R}_n$ , the set of all Artin presentations on  $n$  generators, forms a  
 6 group canonically isomorphic to  $P_n \times \mathbb{Z}^n$ . Note that the group composition law in  
 7 AP theory can be defined purely group theoretically with no mention of braids [20,  
 8 p. 227].

9 Backing up for the moment, the two most immediate objects defined by an  
 10 Artin presentation  $r$  are  $\pi(r)$ , the group presented by  $r$ , and  $A(r)$ , the exponent  
 11 sum matrix of  $r$ . The  $n \times n$  integer matrix:

$$[A(r)]_{ij} = \text{exponent sum of } x_i \text{ in } r_j$$

13 is a presentation matrix of the abelianization of  $\pi(r)$ . Further,  $A(r)$  is the linking  
 14 matrix of  $\beta(r)$ , and hence is symmetric. There are several ways to prove  $A(r)$  is  
 15 symmetric: braids and linking, the symplectic property of closed surface homeo-  
 16 morphisms [20], and the new proof using combinatorial group theory [6].

17 An Artin presentation  $r$  determines a unique closed, orientable 3-manifold,  
 18  $M^3(r)$ , by Winkelnkemper's open book construction. This open book has planar  
 19 page  $\Omega_n$ , homeomorphism  $h(r)$  and binding  $\partial\Omega_n \times 0$ . The fundamental group of  
 20  $M^3(r)$  is isomorphic to  $\pi(r)$ . Hence,  $A(r)$  is a presentation matrix of the first inte-  
 21 gral homology group of  $M^3(r)$ , and  $M^3(r)$  is an integral homology 3-sphere if and  
 22 only if  $\det A(r) = \pm 1$ .

23 Every closed, orientable 3-manifold is homeomorphic to some  $M^3(r)$ . This is a  
 24 fundamental result of González-Acuña [12]. Therefore, *Artin presentations charac-*  
 25 *terize the fundamental groups of closed, orientable 3-manifolds.* Open book decom-  
 26 positions of 3-manifolds yield simple proofs of important theorems and new results  
 27 in 3-manifold theory [17, p. 617], including Bing's characterization of the 3-sphere,  
 28 and the existence of codimension-1 foliations and contact forms.

29 An Artin presentation  $r$  also determines  $W^4(r)$ , a unique smooth, compact, sim-  
 30 ply connected 4-manifold with boundary, where  $\partial W^4(r) = M^3(r)$ . The 4-manifold  
 31  $W^4(r)$  is given by a relative open book construction by extending  $h(r)$  to all of  
 32  $S^2$  by the identity [20, p. 250]. Equivalently,  $W^4(r)$  is the 2-handlebody where  $n$   
 33 2-handles are attached to  $D^4$  according to the closure of the framed, pure braid  
 34  $\beta(r) \subset S^3 = \partial D^4$  [7, Sec. 2.1]. The original construction by Winkelnkemper was  
 35 motivated by his discovery of the symmetry of  $A(r)$  for Artin presentations. Note  
 36 that  $A(r)$  represents the quadratic form of  $W^4(r)$ .

37 While all closed, orientable 3-manifolds appear in AP theory, it is unknown  
 38 which 4-manifolds appear. The main complexity of a 4-manifold is due to its  
 39 2-handles. Still, certain Mazur manifolds, with nonempty boundary a nontrivial  
 40 homology 3-sphere, require a 1-handle, and therefore do not appear as  $W^4(r)$ . If  
 41 the boundary of  $W^4(r)$  is  $S^3$ , then it is natural to close up with a 4-handle and say  
 that  $r$  determines a smooth, *closed*, simply connected 4-manifold  $W^4(r) \cup_{\partial} D^4$ . It

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1 is not known whether all smooth, *closed*, simply connected 4-manifolds appear in  
 2 this way. Many interesting closed 4-manifolds do appear in AP theory: all elliptic  
 3 surfaces  $E(n)$ , in particular the Kummer surface  $K3 = E(2)$  (see [7]). Of course,  
 4 any smooth, closed, simply connected 4-manifold that requires a 1-handle cannot  
 5 appear in AP theory; currently no such manifold is known to exist (see [11, p. 344]).

AP theory has a natural knot and linking theory that is skein free, purely group  
 theoretic, and functorial with respect to the Torelli action as described below. This  
 knot and linking theory is at once canonical and sufficiently general. There are  $n+1$   
 distinguished knots,  $k_0, k_1, \dots, k_n$  in  $M^3(r)$  given by the binding  $\partial\Omega_n \times 0$  in the  
 open book construction. The knot groups  $G_i$  of the knots  $k_i$  are presented by (see  
 [20, pp. 226–227] and [7, Sec. 2.1]):

$$G_0 = \langle x_1, \dots, x_n \mid r_1 = r_2 = \dots = r_n \rangle,$$

$$G_i = \langle x_1, \dots, x_n \mid r_1, r_2, \dots, r_{i-1}, r_{i+1}, \dots, r_n \rangle, \quad i \neq 0.$$

Moreover, if  $M^3(r)$  is an integral homology 3-sphere, then the peripheral structures  
 7  $m_i, l_i$  of the knots  $k_i$  are given by:  $m_0 = \text{any } r_i$ ,  $l_0 = x_1 x_2 \dots x_n m_0^{-s}$  where  $s$   
 8 is the sum of the entries in  $A(r)^{-1}$ , and for  $i \neq 0$ ,  $m_i = r_i$ ,  $l_i = x_i m_i^{-b_i}$  where  
 9  $b_i = [A(r)^{-1}]_{ii}$ .

The generality of this linking theory is exemplified by: *if  $L$  is any link in*  
 11 *any closed, orientable 3-manifold  $M^3$ , then the pair  $(M^3, L)$  is homeomorphic to*  
 12  *$(M^3(r), K)$  for some Artin presentation  $r$ , where  $K$  is the sublink  $k_1, \dots, k_m$  of the*  
 13 *binding. This result is due to González-Acuña<sup>a</sup> (unpublished, see [5] for a proof).*  
 Notice that one recovers González-Acuña's fundamental result, *every closed, ori-*  
 15 *entable 3-manifold is an  $M^3(r)$* , by taking  $L$  to be the empty link.

An important theme in AP theory is that topological invariants of the 3- and  
 17 4-manifolds  $M^3(r)$  and  $W^4(r)$  should be computed group theoretically solely in  
 function of the discrete Artin presentation  $r$ . All homological information is deter-  
 19 mined by  $A(r)$ . González-Acuña's formula for the Rohlin invariant of an integral  
 homology 3-sphere  $M^3(r)$  is more interesting. For clarity, assume<sup>b</sup>  $A(r) = I$ . Let  
 21  $\Delta$  be the Alexander polynomial of the associated presentation:

$$\langle x_1, \dots, x_n \mid x_i r_i = r_i x_{i+1}, \quad i = 1, \dots, n-1 \rangle$$

23 and let  $d = \Delta(-1)$ . Then

$$\mu(M^3(r)) = \frac{d^2 - 1}{8} \pmod{2}.$$

25 Generalizing González-Acuña's formula is a group theoretic formula for the  
 Casson invariant of a rational homology 3-sphere  $M^3(r)$  [5]. Again, for clarity

<sup>a</sup>The author thanks González-Acuña for fruitful discussions concerning this result.

<sup>b</sup>It is open whether every integral homology 3-sphere is represented by an Artin presentation with  $A(r) = I$ . The analogue for Heegaard decompositions is true.

1 assume  $A(r) = I$ . For  $i = 1, \dots, n$ , let  $H_i$  be the associated presentation:

$$H_i = \langle x_1, \dots, x_n \mid r_1, \dots, r_{i-1}, x_{i+1}, \dots, x_n \rangle$$

3 and let  $\Delta_i$  denote the Conway normalized Alexander polynomial of the group pre-  
 5 sented by  $H_i$  (i.e.  $\Delta_i(1) = 1$  and  $\Delta_i(t) = \Delta_i(t^{-1})$ ). Recall that  $\Delta_i$  can be computed  
 group theoretically in function of  $H_i$  (which is in function of  $r$ ) using the Fox free  
 calculus and a computer algebra system such as MAGMA. Then

$$7 \quad \lambda(M^3(r)) = \frac{1}{2} \sum_{i=1}^n \Delta_i''(1).$$

Simple examples show that the Casson invariant is not simply the integer  $(d^2 - 1)/8$   
 9 in González-Acuña's formula for the Rohlin invariant. The general formula for the  
 Casson invariant of a rational homology 3-sphere  $M^3(r)$  when  $A(r) \neq I$  requires  
 11 correction terms determined by the Alexander polynomials of the knots  $k_i$  [5].

This already shows that all 3D Seiberg–Witten invariants can be computed  
 13 *group theoretically* in AP theory since such invariants are determined by  $\lambda$  [15]. The  
 analogous 4D problem is open (see Turaev [19, p. viii]). We conjecture that a strong  
 15 duality holds between the AP theoretic linking theory in the boundary 3-manifold  
 $M^3(r)$  and the smooth geometry of the bounded 4-manifold  $W^4(r)$ . Smooth invari-  
 17 ants should be in function of group theoretic knot invariants. Relationships between  
 smooth invariants and Alexander polynomials of knots have already surfaced [8].  
 19 See Sec. 6 for further discussion.

The group structure on  $\mathcal{R}_n$ , the set of Artin presentations on  $n$  generators, is  
 21 fundamental. If  $r, r' \in \mathcal{R}_n$  and  $\cdot$  denotes the group operation in  $\mathcal{R}_n$ , then  $r' \cdot r$   
 is the composite Artin presentation. Furthermore,  $A$  is a group homomorphism  
 23  $\mathcal{R}_n \rightarrow \text{Mat}_n(\mathbb{Z})$ , namely

$$A(r' \cdot r) = A(r') + A(r).$$

25 Artin presentations whose exponent sum matrices are identically zero are called  
*Torelli*. The Torelli in  $\mathcal{R}_n$  form a subgroup of  $\mathcal{R}_n$  canonically isomorphic to  $[P_n, P_n]$ ,  
 27 the commutator subgroup of the  $n$  strand pure braid group  $P_n$ . The classical Torelli  
 group consists of elements of the mapping class group of the closed, oriented genus  
 29  $n$  surface that act trivially on homology. The Torelli in AP theory form a subgroup  
 of the classical Torelli group.

31 Two Artin presentations  $r, r' \in \mathcal{R}_n$  differ by multiplication by a Torelli if and  
 only if  $A(r) = A(r')$ . Also,  $A(t \cdot r) = A(r)$  for all Torelli  $t$ , so composing an  
 33 Artin presentation with a Torelli preserves the integer homology of both  $M^3(r)$  and  
 $W^4(r)$ . This action of the Torelli subgroup on  $\mathcal{R}_n$  by left translation is extremely  
 35 rich and complicated. It can change or preserve different properties of the 3- and  
 4-manifolds and the knots  $k_i$ .

37 A main purpose of this paper is to show that the Torelli action can preserve the  
 continuous topology of  $W^4(r)$  while changing its smooth topological type.

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1 Let  $r$  be an Artin presentation such that  $\det A(r) = \pm 1$ . Let  $t$  be a Torelli such  
 2 that  $M^3(r)$  and  $M^3(t \cdot r)$  are orientation preserving homeomorphic. Then, since  
 3  $A(r) = A(t \cdot r)$ , the 4-manifolds  $W^4(r)$  and  $W^4(t \cdot r)$  will be homeomorphic by  
 4 Freedman's theorem (see [9] and [11, p. 448]). The question arises whether these  
 5 4-manifolds are necessarily diffeomorphic.

Using work of Akbulut [1, 2], we show that

7 **Theorem 2.1.** *There exists Artin presentations  $r \in \mathcal{R}_n$  and Torelli  $t \in \mathcal{R}_n$  for all*  
 *$n \geq 10$  such that  $W^4(r)$  and  $W^4(t \cdot r)$  are homeomorphic but not diffeomorphic.*

9 The common boundary of these 4-manifolds is the simplest hyperbolic integral  
 10 homology 3-sphere, namely the  $1/2$  Dehn sphere of the figure eight knot in  $S^3$  (see  
 11 Sec. 4 below).

12 Thus, smooth structures on an underlying topological 4-manifold can be changed  
 13 in a general, exterior, purely group theoretic manner by a canonical action of the  
 14 Torelli subgroup. This is different from internal surgery methods. Here, pure group  
 15 theory generates new 4D smooth structures.

16 These examples are the first step and they show that the group theoretic Torelli  
 17 action can change the *smooth* topology of 4-manifolds. The bigger picture involves  
 18 computing smooth invariants (Donaldson, Seiberg–Witten, and so forth) of the  
 19 4-manifolds  $W^4(r)$  group theoretically in function of  $r$ . This would provide a more  
 20 general framework to study 4-manifolds and the Torelli action in AP theory (see  
 21 Sec. 6).

22 An intriguing question arises, further discussion of which we defer to other  
 23 papers. The *global* consequences of solving the 4D quantum Yang–Mills mass gap  
 24 “Millennium” problem [14, p. 6] are closely related to the behavior of Donaldson  
 25 invariants of algebraic surfaces [22, p. 25]. Generalizing Witten's work [21] on this  
 26 subject from the Kähler case to the general case involves serious *analytical* obstruc-  
 27 tions. Developing purely group theoretic Donaldson invariants could be a promising  
 attack.

### 29 3. Constructing the Artin Presentations

30 Figure 1 contains two framed, pure braids  $s_1$  and  $s_2$  on ten strands. A framed  
 31 pure braid determines an Artin presentation [7], so let  $s_1$  and  $s_2$  denote the Artin  
 32 presentations in  $\mathcal{R}_{10}$  corresponding to these framed pure braids, where no confusion  
 33 should arise.

34 **Theorem 3.1.** *The Artin presentations  $s_1$  and  $s_2$  differ by multiplication*  
 35 *by a Torelli  $t$ . Furthermore,  $W^4(s_1)$  and  $W^4(s_2)$  are homeomorphic but not*  
*diffeomorphic.*

36 The proof is below. First, note some properties of these Artin presentations.  
 37 The matrices  $A(s_1)$  and  $A(s_2)$  are equal and have determinant  $+1$  (see Fig. 2).

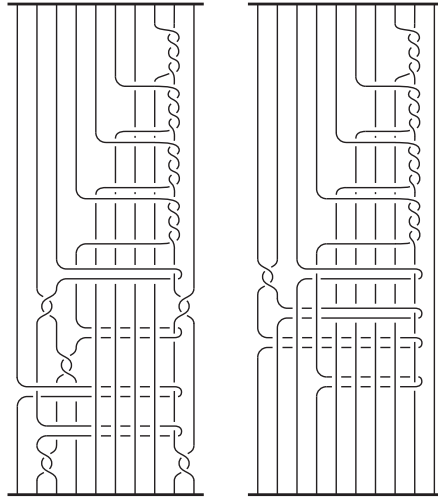


Fig. 1. Pure braids  $s_1$  and  $s_2$ , each with framings  $-1, -2, -1, -2, -1, -1, -1, -1, -23, -1$  from left to right.

$-1$	$-1$									$-1$
$-1$	$-2$									$-1$
		$-1$	$-1$							$-1$
		$-1$	$-2$							$+1$
				$-1$						$-2$
					$-1$					$-2$
						$-1$				$-2$
							$-1$	$-2$		$-2$
$-1$	$-1$	$-1$	$+1$	$-2$	$-2$	$-2$	$-2$	$-23$		$-1$
										$-1$

Fig. 2. Matrix  $A(s_1) = A(s_2)$  with  $\det = +1$ .

1 Hence,  $M^3(s_1)$  and  $M^3(s_2)$  are integral homology 3-spheres and  $s_1$  and  $s_2$  differ  
 2 by multiplication by the Torelli  $t = s_2 \cdot s_1^{-1}$ . The fact that  $M^3(s_1)$  and  $M^3(s_2)$   
 3 are orientation preserving homeomorphic follows from work of Akbulut [1, 2] along  
 4 with the construction of these pure braids below.

5 In Sec. 4, the 3-manifolds  $M^3(s_1)$  and  $M^3(s_2)$  are identified as the simplest  
 6 hyperbolic integral homology 3-sphere, namely the  $1/2$  Dehn sphere of the figure  
 7 eight knot of  $S^3$ .

8 Freedman's theorem, as discussed above, implies  $W^4(s_1)$  and  $W^4(s_2)$  are homeo-  
 9 morphic. Below it is shown that they are not diffeomorphic. This remains true  
 10 stably, that is upon finitely many blow ups with  $\overline{\mathbb{C}P^2}$ . Hence, one obtains similar  
 11 examples in  $\mathcal{R}_n$  for all  $n \geq 10$ .

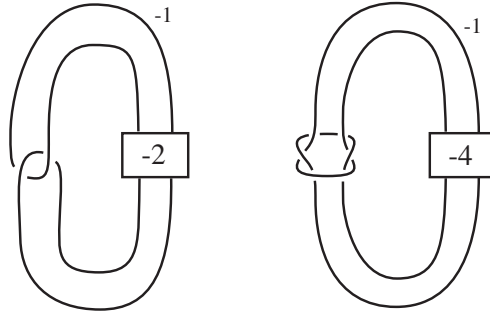


Fig. 3. Two 4-manifolds  $\overline{Q}_1$  (left) and  $\overline{Q}_2$  (right).

1 The construction of the Artin presentations  $s_1$  and  $s_2$  begins with the two  
 2 manifolds  $\overline{Q}_1$  and  $\overline{Q}_2$  in Fig. 3. These interesting manifolds were discovered by  
 3 Akbulut [1, 2]. In particular, by reversing each crossing and changing the sign of the  
 4 framings from  $-1$  to  $+1$ , one obtains Akbulut's manifolds  $Q_1$  and  $Q_2$  respectively  
 5 (see [2, p. 357]). As usual,  $\overline{M}$  denotes the oriented manifold obtained from the  
 6 oriented manifold  $M$  by reversing the orientation on every component.

7 Akbulut showed [2] that  $Q_1$  and  $Q_2$  are homeomorphic but not diffeomorphic.  
 8 Hence,  $\overline{Q}_1$  and  $\overline{Q}_2$  are homeomorphic but not diffeomorphic. His proof relies on  
 9 the computation of a Donaldson invariant [1]. This result also follows from the  
 10 adjunction inequality [11, pp. 448–449].

11 For present purposes, a stable version of this result is needed. This was already  
 12 implicit in Akbulut's work.<sup>c</sup>

13 **Claim 3.2.**  $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2$  and  $\overline{Q}_2 \#_k \overline{\mathbb{C}P}^2$  are homeomorphic but not diffeomorphic  
 14 for all  $k \geq 0$ .

15 **Proof.** Following Akbulut's notation [2], the manifolds in question are homeomor-  
 16 phic since  $\overline{Q}_1$  and  $\overline{Q}_2$  are homeomorphic. Akbulut constructed a smooth, compact,  
 17 connected, simply connected 4-manifold  $M_1$  with  $\partial M_1 = \partial Q_1 = \partial Q_2$  [1]. Fur-  
 18 thermore,  $Q_2$  splits off a  $\mathbb{C}P^2$  summand, that is  $Q_2 = W_1 \# \mathbb{C}P^2$  where  $W_1$   
 19 is a smooth, compact, contractible 4-manifold with  $\partial W_1 = \partial Q_2$  [2, p. 359]. Thus,  
 $\overline{Q}_2 = \overline{W}_1 \# \overline{\mathbb{C}P}^2$  splits off a  $\overline{\mathbb{C}P}^2$ .

Suppose  $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2$  and  $\overline{Q}_2 \#_k \overline{\mathbb{C}P}^2$  are diffeomorphic for some  $k \geq 0$ . Since  $\overline{Q}_2 =$   
 $\overline{W}_1 \# \overline{\mathbb{C}P}^2$ , it follows that there are  $k + 1$  disjoint smoothly embedded 2-spheres in  
 $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2$ , each with self intersection number  $-1$ . Thus,  $\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2 = \overline{V} \#_{k+1} \overline{\mathbb{C}P}^2$   
 for some smooth, contractible 4-manifold  $V$  with  $\partial V = \partial Q_1$  (see [11, p. 46]). Let

<sup>c</sup>This also follows from uniqueness of minimal models of surfaces of general type, as pointed out  
 to the author by Gompf.



$\widetilde{M} = M_1 \cup_{\partial} \overline{Q}_1$  and  $M' = M_1 \cup_{\partial} \overline{V}$ . Then

$$\begin{aligned} \widetilde{M} \#_k \overline{\mathbb{C}P}^2 &= (M_1 \cup_{\partial} \overline{Q}_1) \#_k \overline{\mathbb{C}P}^2 \\ &= M_1 \cup_{\partial} (\overline{Q}_1 \#_k \overline{\mathbb{C}P}^2) \\ &= M_1 \cup_{\partial} (\overline{V} \#_{k+1} \overline{\mathbb{C}P}^2) \\ &= M' \#_{k+1} \overline{\mathbb{C}P}^2. \end{aligned}$$

1 This is a contradiction (see [2, p. 358], Property 2) and the claim follows.  $\square$

3 The remainder of the proof of the main theorem consists of blowing up  $\overline{Q}_1$  and  $\overline{Q}_2$   
 with finitely many  $\overline{\mathbb{C}P}^2$ s and using isotopy to obtain the closure of two pure braids  
 with equal linking matrices. It is not difficult to blow up a knot or link and apply  
 5 isotopy and handle slides to obtain a pure link. The difficulty lies in doing this to  
 two different links with the ultimate goal of obtaining *equal* linking matrices.

7 Below,  $\overline{Q}_1$  and  $\overline{Q}_2$  are each blown up with 9  $\overline{\mathbb{C}P}^2$ s which, along with isotopy,  
 produces the closures of the pure braids in Fig. 1.

9 Begin by blowing up  $\overline{Q}_1$  from Fig. 3 to obtain the first diagram in Fig. 4. The  
 rest of the figure modifies  $\overline{Q}_1 \# \overline{\mathbb{C}P}^2$  by isotopy. Blowing up again produces the first  
 11 diagram in Fig. 5. The rest of the figure modifies  $\overline{Q}_1 \# 2\overline{\mathbb{C}P}^2$  by isotopy.

13 Blow up twice to obtain the first and second diagrams in Fig. 6. By an isotopy  
 of the second diagram in Fig. 6, one obtains the first diagram in Fig. 7. Another  
 isotopy yields the second diagram in Fig. 7. Figure 8 is obtained by blowing up.

15 Now, an operation is described to blow up and eliminate twists. The first dia-  
 gram in Fig. 9 represents a local picture of a single framed knot. The top two free  
 17 strands connect elsewhere, the bottom two strands connect elsewhere, and the fram-  
 ing coefficient equals  $d$ . The diagram will only be changed locally. The box with  $-1$   
 19 in it represents a single twist, as shown by the second diagram in Fig. 9. Blowing  
 up yields the third diagram, and a local isotopy produces the final diagram. The  
 21 framing  $d$  changes to  $d - 4$ .

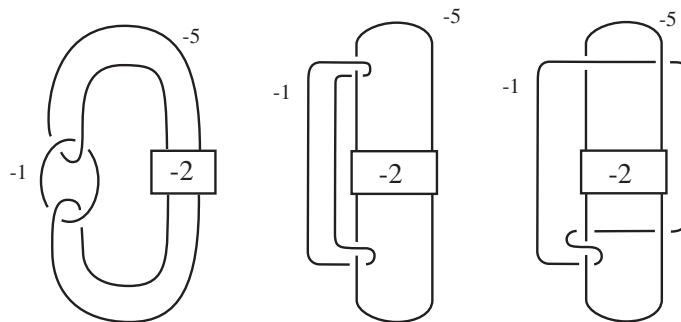
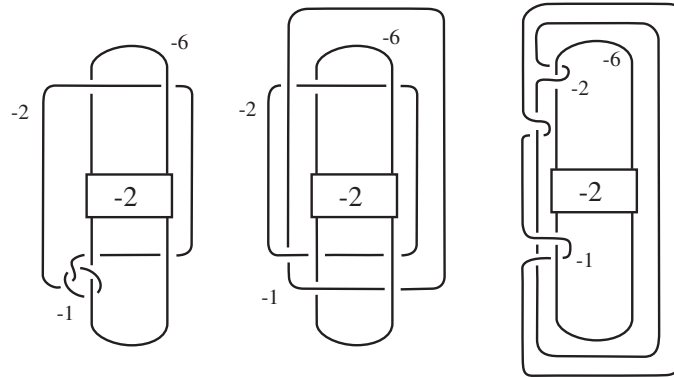
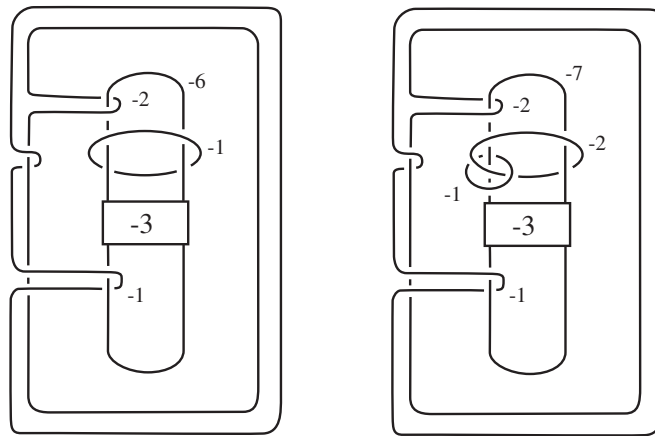


Fig. 4. Two isotopies of  $\overline{Q}_1 \# \overline{\mathbb{C}P}^2$ .

Fig. 5. Blow up of  $\overline{Q}_1 \# \overline{\mathbb{C}P}^2$  and two isotopies.Fig. 6. Two blow ups of  $\overline{Q}_1 \# 2\overline{\mathbb{C}P}^2$ .

1 It is useful to see diagrammatically how to apply this operation and then perform  
 2 isotopy to obtain pure links. Figure 10 shows how to remove multiple twists by  
 3 blowing up. Note that the thickened lines represent parts of the link diagram that  
 4 do not change. Figure 11 shows how to isotop the second diagram in Fig. 10 to put  
 5 the new components in pure link form (framings are unchanged).

6 One obtains the pure braid  $s_1$  in Fig. 1 as follows. The  $-1$  framed circle  $C$  (just  
 7 above the “ $-4$ ” box) in Fig. 8 should be thought of as lying in the upper thickened  
 8 line in Fig. 10 that cuts across the shown knot twice. Perform the operations shown  
 9 in Figs. 10 and 11, and then perform the operation in Fig. 12 on the  $-1$  framed  
 10 circle  $C$  just described. Now, take the portion of the link that was in the upper  
 11 thickened line and slide it up and all the way around to the bottom of the diagram.  
 This produces the pure braid  $s_1$ .

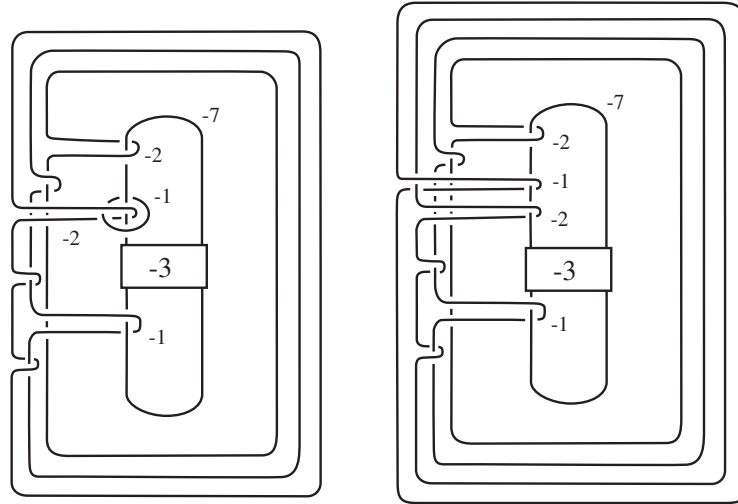


Fig. 7. Two isotopies of  $\overline{Q}_1 \# 4\overline{CP}^2$ .

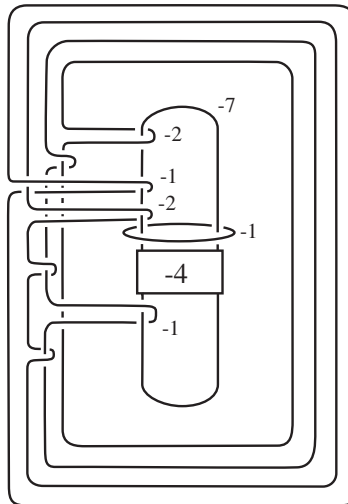


Fig. 8. Blow up of  $\overline{Q}_1 \# 4\overline{CP}^2$ .

1 Proceeding to  $\overline{Q}_2$ , blow up  $\overline{Q}_2$  from Fig. 3 twice to obtain the first diagram  
in Fig. 13. Perform a simple isotopy and then blow up again to obtain the rest of  
3 Fig. 13. Figures 14 and 15 contain straightforward isotopies.

5 The first diagram in Fig. 16 is obtained by isotopy, and the second by blowing  
up. Another isotopy yields Fig. 17. Perform the operations in Figs. 10 and 11 on the  
framed link in Fig. 17. Take the portion of the link that was in the upper thickened

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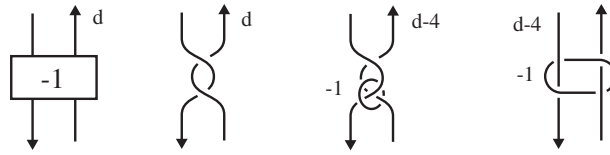


Fig. 9. Blowing up to remove a twist in a single component.

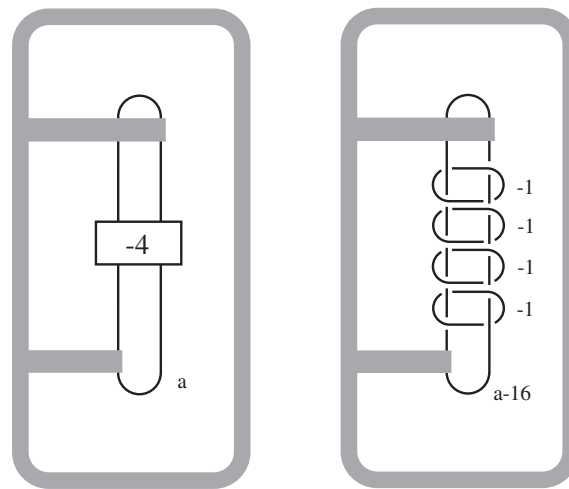


Fig. 10. Four blow ups to remove four twists.

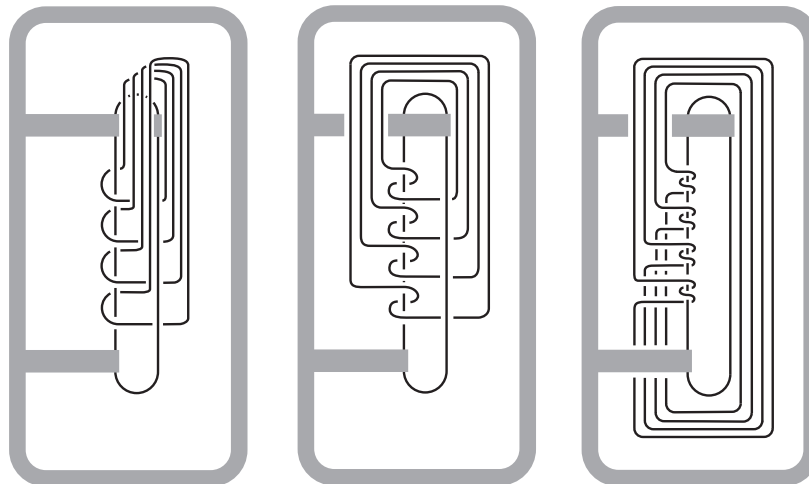


Fig. 11. Isotopy of untwist operation to braid.

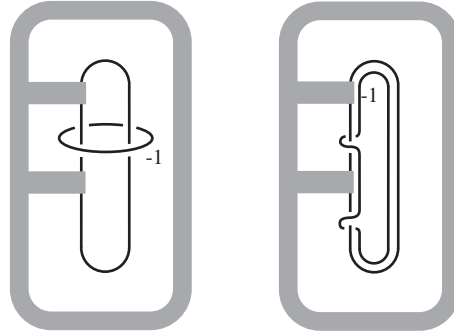


Fig. 12. Isotopy of loop to braid.

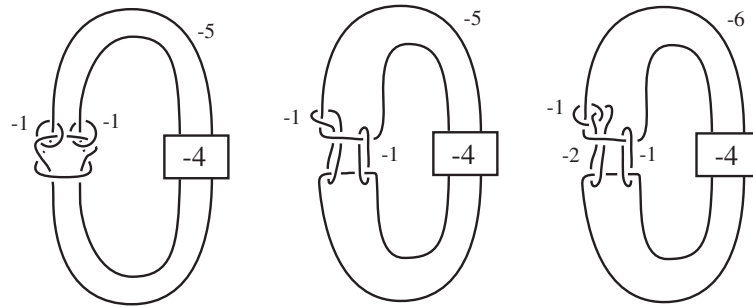


Fig. 13. Isotopy of  $\overline{Q}_2 \# 2\overline{CP}^2$  followed by blow up.

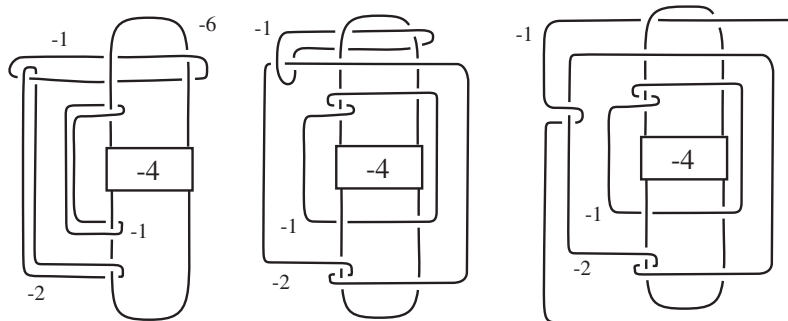


Fig. 14. Isotopies of  $\overline{Q}_2 \# 3\overline{CP}^2$ .

1 line in the operation in Figs. 10 and 11 and slide it up and all the way around to  
 2 the bottom of the diagram. Blow up once more and leave this as the trivial tenth  
 3 strand not linking anything. This produces the second pure braid  $s_2$  in Fig. 1.

4 This completes the construction of the Artin presentations  $s_1$  and  $s_2$  and the  
 5 proof of the main theorem.

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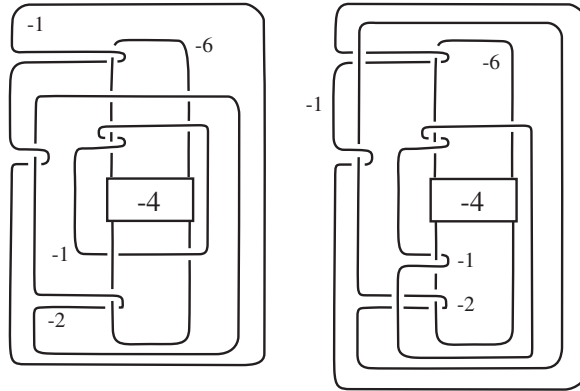


Fig. 15. Isotopies of  $\overline{Q}_2 \# 3\overline{CP}^2$ .

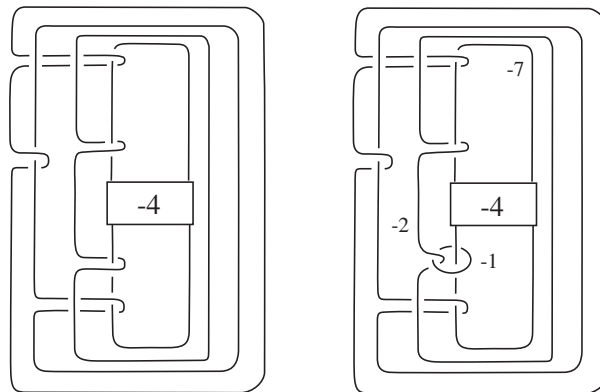


Fig. 16. Isotopy of  $\overline{Q}_2 \# 3\overline{CP}^2$  followed by blow up.

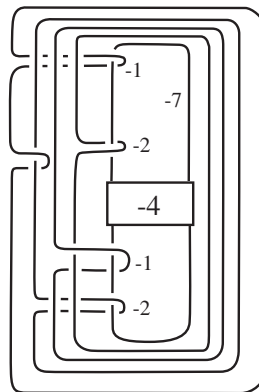


Fig. 17. Isotopy of  $\overline{Q}_2 \# 4\overline{CP}^2$ .

1 **4. Identifying Boundaries**

3 This section shows that the 3-manifolds  $M^3(s1)$  and  $M^3(s2)$  are both homeomor-  
 5 phic to the 1/2 Dehn sphere of the figure eight knot of  $S^3$ , the simplest hyperbolic  
 7 integral homology 3-sphere. It suffices to identify  $\partial\overline{Q}_1$  from Fig. 3 as this Dehn  
 sphere, since the 3-manifolds  $M^3(s1)$ ,  $M^3(s2)$ ,  $\partial\overline{Q}_1$ , and  $\partial\overline{Q}_2$  are all homeomor-  
 phic. One may blow up with  $\overline{\mathbb{C}P}^2$  and/or  $\mathbb{C}P^2$  now since the only concern is the  
 3-manifolds.

9 As in Fig. 18, blow up  $\overline{Q}_1$  with two  $\mathbb{C}P^2$ s to remove the two twists, then blow up  
 with  $\overline{\mathbb{C}P}^2$  as shown. Slide the 0 framed component over the +1 framed component in  
 11 a very simple way (use a trivial band) to obtain the first diagram in Fig. 19. Blow  
 down the right +1 framed component to obtain the second diagram in Fig. 19.  
 13 The third diagram in Fig. 19 is obtained by interchanging the 0 and -1 framed  
 components by isotopy, exactly as one does with a Whitehead link. Take the third  
 15 diagram in Fig. 19, rotate it 90 degrees clockwise and blow down the -1 framed  
 component to introduce a twist. The result is the first diagram in Fig. 20. Finally,  
 17 perform a slam dunk (see [11, pp. 163-164]) to obtain 1/2 surgery on the figure  
 eight knot (Fig. 20), as desired.

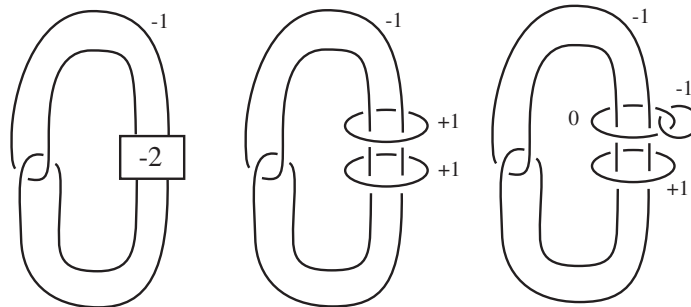


Fig. 18. Blow ups of  $\overline{Q}_1$ , first with two  $\mathbb{C}P^2$ s, then with one  $\overline{\mathbb{C}P}^2$ .

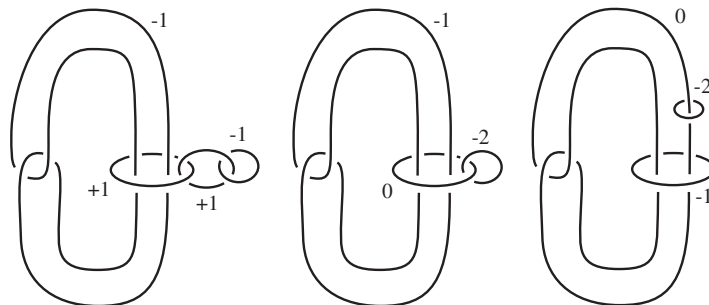


Fig. 19. A 2-handle slide, a blow down (of right +1), and an isotopy.

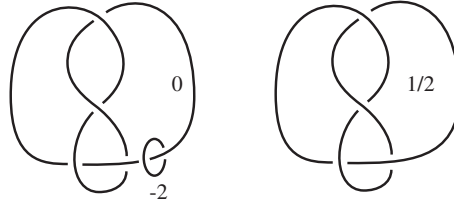
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Fig. 20. Blow down and slam dunk.

1 **5. Knots**

3 The Alexander polynomials of the knots  $k_i(s_1)$  and  $k_i(s_2)$  in the 3-manifolds  $M^3(s_1)$  and  $M^3(s_2)$  are easily found using a computer algebra system such as MAGMA. The Alexander polynomials of the knots in  $M^3(s_1)$  are:

Knot	Alexander Polynomial
$k_1$	$(t^2 - 3t + 1)(t^2 - t + 1)$
$k_2$	$t^2 - 3t + 1$
$k_3$	$2t^{12} - 2t^{11} + t^9 - t^8 + t^6 - t^4 + t^3 - 2t + 2$
$k_4$	$2t^6 - 2t^5 + t^4 - t^3 + t^2 - 2t + 2$
$k_5$	$2t^6 - 2t^5 + t^3 - 2t + 2$
$k_6$	$2t^6 - 2t^5 + t^3 - 2t + 2$
$k_7$	$2t^6 - 2t^5 + t^3 - 2t + 2$
$k_8$	$2t^6 - 2t^5 + t^3 - 2t + 2$
$k_9$	$2t^2 - 3t + 2$
$k_{10}$	$t^2 - 3t + 1$

5

The Alexander polynomials of the knots in  $M^3(s_2)$  are:

Knot	Alexander Polynomial
$k_1$	$t^6 - 6t^5 + 11t^4 - 11t^3 + 11t^2 - 6t + 1$
$k_2$	$3t^2 - 7t + 3$
$k_3$	$t^{18} - t^{17} + 5t^{12} - 8t^{11} + 3t^{10} + t^9 + 3t^8 - 8t^7 + 5t^6 - t + 1$
$k_4$	$t^{10} - t^9 + 5t^6 - 9t^5 + 5t^4 - t + 1$
$k_5$	$(t^2 - t + 1)(t^4 - t^2 + 1)^2$
$k_6$	$(t^2 - t + 1)(t^4 - t^2 + 1)^2$
$k_7$	$(t^2 - t + 1)(t^4 - t^2 + 1)^2$
$k_8$	$(t^2 - t + 1)(t^4 - t^2 + 1)^2$
$k_9$	$(t^2 - t + 1)^2$
$k_{10}$	1

7



Curiously, the knots  $k_0(s1)$  and  $k_0(s2)$  both have Alexander polynomials of degree 108 (when normalized so the lowest degree term is  $t^0$ ). However, the Alexander polynomial of  $k_0(s1)$  is irreducible while that of  $k_0(s2)$  factors into the product of:

$$\begin{aligned} &t^2 - t + 1, \\ &t^6 - t^3 + 1, \\ &t^8 - t^7 + t^5 - t^4 + t^3 - t + 1, \\ &t^8 + t^7 - t^5 - t^4 - t^3 + t + 1, \\ &(t^{18} - t^9 + 1)^2, \\ &t^{24} - t^{21} + t^{15} - t^{12} + t^9 - t^3 + 1, \text{ and} \\ &t^{24} + t^{21} - t^{15} - t^{12} - t^9 + t^3 + 1. \end{aligned}$$

1 The only knots whose *groups* we recognize are  $k_9(s1)$  and  $k_{10}(s2)$ . The group of  
 3  $k_9(s1)$  is isomorphic to the group of the  $5_2$  knot in  $S^3$  with Alexander polynomial  
 5  $2t^2 - 3t + 2$ . It is easy to see that  $k_{10}(s2)$  is the trivial knot in  $M^3(s2)$  from the  
 braid  $s2$ . The Torelli  $t$  takes the knot  $k_9(s1)$  to the knot  $k_9(s2)$ , the latter of which  
 has a huge presentation and Alexander polynomial  $(t^2 - t + 1)^2$ .

The lengths of the individual relations in  $s1$  and  $s2$  are:

Relation	$s1$	$s2$	Relation	$s1$	$s2$
1	3187	1723	6	269	1733
2	13506	734	7	251	1715
3	8103	243	8	245	1709
4	7132	5624	9	7475	8215
5	323	1787	10	4891	1

7

Thus,  $s1$  has total relator length 45382 and  $s2$  has total relator length 23484.  
 9 Notice that even though  $s2$  splits off a  $\overline{\mathbb{C}P^2}$  summand, it is the manifold with  
 nontrivial Donaldson invariants [1, 2]. It seems curious that  $s2$  is the tighter pre-  
 11 sentation. The Kummer surface  $K3 = E(2)$  also has nontrivial smooth invariants  
 and one Artin presentation for  $K3$  in  $\mathcal{R}_{22}$  has total relator length 4562 [7].

13 Finally, the inverse matrix  $A(s1)^{-1}$  in Fig. 21 gives the peripheral structures of  
 the knots  $k_i$  as described in Sec. 2.

## 15 6. Future Direction

The first basic questions in AP theory concern representing 3- and 4-manifolds.  
 17 All closed, orientable 3-manifolds appear. Still, it is open whether every integral  
 homology 3-sphere can be represented by an Artin presentation  $r$  with  $A(r) = I$ ; the  
 19 analogue for Heegaard decompositions is true. More generally, can every rational  
 homology 3-sphere can be represented by an Artin presentation  $r$  with  $A(r)$  a  
 21 diagonal matrix?

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$$\begin{array}{cccccccccc}
-3 & 1 & -3 & 2 & -2 & -2 & -2 & -2 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & -11 & 7 & -6 & -6 & -6 & -6 & 3 & 0 \\
2 & 0 & 7 & -5 & 4 & 4 & 4 & 4 & -2 & 0 \\
-2 & 0 & -6 & 4 & -5 & -4 & -4 & -4 & 2 & 0 \\
-2 & 0 & -6 & 4 & -4 & -5 & -4 & -4 & 2 & 0 \\
-2 & 0 & -6 & 4 & -4 & -4 & -4 & -5 & 2 & 0 \\
-2 & 0 & -6 & 4 & -4 & -4 & -4 & -5 & 2 & 0 \\
1 & 0 & 3 & -2 & 2 & 2 & 2 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}$$

Fig. 21. Matrix  $A(s1)^{-1} = A(s2)^{-1}$ .

1 A more difficult question is: *which closed 4-manifolds appear in AP theory?*  
 Recall that if  $M^3(r) = S^3$ , then it is natural to close up  $W^4(r)$  with a 4-handle  
 3 (which may be done uniquely), thus obtaining a closed, smooth, simply connected  
 4-manifold.

5 Representing closed 4-manifolds in AP theory involves two difficulties. First,  
 one must represent the manifold as a 2-handlebody; it is open whether all closed,  
 7 smooth, simply connected 4-manifolds admit 2-handle decompositions (see [11,  
 p. 344]). Second, one must arrange that the link along which the 2-handles are  
 9 attached is the closure of a pure braid. This second step surely requires birth and  
 death of cancelling handle pairs, which is quite subtle [10].

11 The empty presentation  $r = \langle \rangle$  is the unique Artin presentation in  $\mathcal{R}_0$  and  
 gives the closed 4-manifold  $S^4$ . In  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , Artin presentations are determined  
 13 by their exponent sum matrices and there are no nontrivial Torelli. However, the  
 Torelli subgroup in  $\mathcal{R}_3$  is already infinitely generated and very complicated.

15 The manifolds constructed in this paper show that the Torelli action in  $\mathcal{R}_{10}$   
 can change the smooth structure of a 4-manifold while preserving its continuous  
 17 topological type. One would like to find similar examples where: the boundaries are  
 $S^3$  (closed 4-manifolds), the 4-manifolds are irreducible, and/or  $n = b_2 < 10$ . A  
 19 potential example with closed 4-manifolds has already appeared [7, p. 20]. In this  
 example,  $r \in \mathcal{R}_{22}$  is an Artin presentation for the Kummer surface (so  $\pi(r) = 1$ ) and  
 21 a certain Torelli  $t$  is shown to preserve  $\pi = 1$  while changing *exactly one* Alexander  
 polynomial, namely  $k_0$ . In the above main examples the Alexander polynomials  $k_0$   
 23 also had very interesting properties.

25 Deeper is the study of smooth invariants in AP theory. An Artin presentation  
 is a discrete, purely group theoretic object, yet it provides a very nice geomet-  
 ric decomposition of both the 3- and the 4-manifold. Moreover, the intrinsic knot  
 27 theory seems central to computing smooth invariants (already a fact with the Cas-  
 son invariant). Note that the smooth 4-manifold  $W^4(r)$  is completely determined  
 by the framed pure braid  $\beta(r)$  and the knots  $k_i(r)$  are meridians of this braid.

1 These considerations, along with the fact that knots and Alexander polynomials  
 are already known to be related to smooth 4D invariants [8], lead us to:

3 **Conjecture 3.3.** *In AP theory, the smooth structure of the bounded 4-manifold is  
 dual to the canonical knot theory in the bounding 3-manifold.*

5 This is the AP theoretic analogue of the Gopakumar–Vafa conjecture which states  
 that knot theoretic invariants in certain 3-manifolds correspond to certain counts  
 7 of holomorphic curves in an associated vector bundle [13].

In AP theory, the Alexander polynomials of the knots  $k_i$  in the bounding  
 9 3-manifold  $M^3(r)$ , along with their peripheral structures, should determine non-  
 trivial smooth invariants of the canonically associated bounded smooth 4-manifold  
 11  $W^4(r)$ . The geometric transitions in AP theory result from the Torelli action.

The broader picture of the Torelli action described in this paper runs as fol-  
 13 lows. Start with an Artin presentation  $r$  with  $M^3(r)$ ,  $W^4(r)$ , and  $A(r)$  possessing  
 whichever desired properties. Compose this Artin presentation with many Torelli,  
 15 using a computer algebra system, such as MAGMA, until  $M^3(r)$  and  $M^3(t \cdot r)$  are  
 orientation preserving homeomorphic. Compute a group theoretic smooth invariant  
 17 of  $W^4(t \cdot r)$ , and conclude the smooth structure has been changed.

The first difficulty with this program is recognizing the 3-manifolds. The truth  
 19 of the 3D Poincaré conjecture would reduce closed 4-manifold theory in AP theory  
 to the study of Artin presentations of the trivial group. In some cases, certain  
 21 group theoretic invariants of the fundamental groups of the 3-manifolds (found on  
 the computer) hint that the 3-manifolds may be the same, after which the Kirby  
 23 calculus can be employed to show that they are actually homeomorphic.

The main advantages of AP theory are generality and computability. All  
 25 3-manifolds and links therein appear, and a sufficiently rich collection of 4-manifolds  
 appears (possibly all closed, smooth, simply connected 4-manifolds). Invariants  
 27 of both the discrete Artin presentation  $r$  and the knots  $k_i$  are computed using  
 MAGMA on the computer, and there are no computational difficulties here.

An important question is: how can one utilize the structure of the 3-manifold  
 29 as an open book with *planar* page, and the near mapping tori structure of the  
 4-manifold as a special 2-handlebody to compute 4D gauge theoretic invariants in  
 31 AP theory? We remark that open book decompositions have played an important  
 33 role in understanding contact structures on 3-manifolds.

Finally, the Fintushel–Stern knot surgery operation can be represented as a  
 35 series of  $\pm 1$  generalized logarithmic transformations on null-homologous tori [18,  
 pp. 9–10]. The change in Seiberg–Witten invariants under such transformations is  
 37 well understood [16, 18]. Can the smooth structure changing Torelli action in AP  
 theory be represented (canonically) as a series of such transformations? Compare  
 39 the perturbations required with computing the Casson invariant in AP theory.  
 The perturbations are nontrivial, yet they do not remove one from the realm of AP  
 41 theory [5].

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