Grade School Triangles

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Abstract. In grade school, students learn a standard set of Euclidean triangles. Among this set, the usual 45–45–90 and 30–60–90 triangles are the only right triangles with rational angles and side lengths each containing at most one square root. Are there any other such right triangles? We answer this question completely and present an elegant complement called Ailles' rectangle that deserves to be in every geometry teacher's toolkit.

1. INTRODUCTION. During grade school, one learns about the Euclidean triangles shown in Figure 1. These include: (1) 60–60–60, 45–45–90, and 30–60–90 triangles with nice angles and side lengths and (2) the infinite collection of Pythagorean triple triangles. As no other triangles are nearly as widely known, one might assume these are the only nice ones. This sentiment is almost true. To make this precise, we need a few definitions.

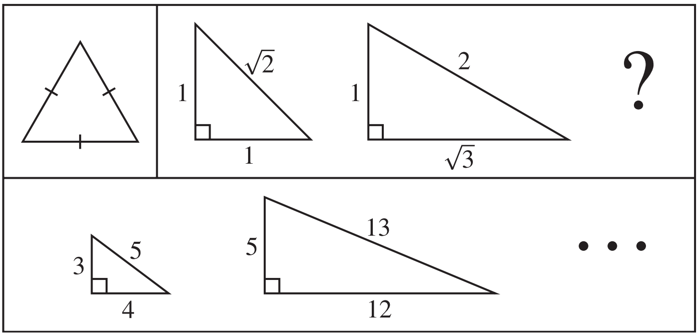

Figure 1. Grade school triangles.

Definitions. An angle is *rational* provided it is commensurable with a straight angle; equivalently, its degree measure is rational or its radian measure is a rational multiple of π. A *quadratic irrational* is a number of the form $r + s\sqrt{d}$ where r and s are rational, $s \neq 0$, and $d \notin \{0, 1\}$ is a squarefree integer (i.e., $p^2 \nmid d$ for all primes $p \in \mathbb{Z}$). A line segment is *rational* or *quadratic irrational* provided its length is rational or quadratic irrational respectively.

Before we state our main result, let us put it into context by describing a sense in which each triangle shown in row one of Figure 1 is unique. This information is contained in Table 1.

These three characterizations are simple corollaries of the following well-known fact (see [**10**, p. 41] for a proof of Fact 1).

Fact 1. *The only rational values of the circular trigonometric functions at rational multiples of* π *are the obvious ones, namely* 0, $\pm 1/2$, *and* ± 1 *for cosine and sine,* 0 *and* ± 1 *for tangent and cotangent, and* ± 1 *and* ± 2 *for secant and cosecant.*

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Table 1. Uniqueness of 60–60–60, 45–45–90, and 30–60–90 triangles.

Triangle Properties	Unique Similarity Type
Rational angles Rational sides	60–60–60
Right triangle Rational angles Rational legs	45–45–90
Right triangle Rational angles Rational hypotenuse At least one rational leg	30–60–90

Proofs of the three characterizations in Table 1 now run as follows. For the first, observe that each angle in such a triangle has rational cosine by the law of cosines and hence has measure $\pi/3$, $\pi/2$, or $2\pi/3$ by Fact 1. The angle sum is π and the characterization follows (see [**6**, pp. 228–229] for an alternative proof). For the second, observe that the tangent of an acute angle is rational and apply Fact 1 (see [**2, 3, 4**] for alternative proofs of this characterization). For the third, observe that the cosine of one of the acute angles is rational and apply Fact 1.

Any one of the three characterizations in Table 1 immediately implies that right triangles with rational angles and rational sides do not exist. In other words, we have the following.

Corollary 1. *The acute angles in each Pythagorean triple triangle are irrational.*

This is why, in grade school, one usually does not learn much about the acute angles in Pythagorean triple triangles! Corollary 1 deserves to be more widely known. For instance, in 2004 Florida Governor Jeb Bush was naturally stumped when a high school student asked him the angles in the (3, 4, 5) triangle. Subsequently, there appeared in the media some bad math, and even some negativity towards mathematics, surrounding this incident. This was quite unfortunate since, in fact, it was a wonderful question with intimate relations to Gaussian integers and planar polygons. See [**3**] for a very natural proof of Corollary 1, accessible to bright high school students, which uses unique factorization of Gaussian integers and presents some applications to geometry.

At this point, it would be a sin of omission not to mention the following enhancement of Corollary 1.

Fact 2. *The acute angles in each Pythagorean triple triangle have transcendental radian measures and transcendental degree measures.*

To prove this fact, let $\triangle ABC$ be as in Figure 2 with positive integer side lengths satisfying $a^2 + b^2 = c^2$. Here α denotes the radian measure of the angle at A. For the first claim, $\alpha = \arctan a/b$ is transcendental by the generalized Lindemann theorem [**10**, Thm. 9.11, p. 131] which states that the exponentials of distinct algebraic numbers are linearly independent over the field of algebraic numbers. The second claim is equivalent to α/π being transcendental. We have

$$\frac{\alpha}{\pi} = \frac{\arccos \frac{b}{c}}{\pi} = \frac{-i \ln\left(\frac{b}{c} + i\sqrt{1 - \frac{b^2}{c^2}}\right)}{\pi} = \frac{\ln\left(\frac{b}{c} + i\frac{a}{c}\right)}{i\pi} = \frac{\ln\left(\frac{b}{c} + i\frac{a}{c}\right)}{\ln(-1)}.$$

The Gelfond-Schneider theorem [**10**, Thm. 10.2, p. 135] states that if $\delta, \epsilon \neq 0$ are algebraic, then values of $\ln \delta / \ln \epsilon$ are either rational or transcendental. Corollary 1 implies that α/π is irrational and Fact 2 follows.

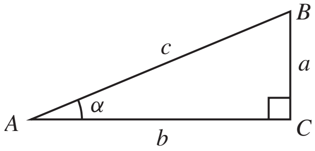

Figure 2. Pythagorean triple triangle $\triangle ABC$.

We now come to the question that provoked the present work, namely: *which right triangles have rational angles and rational or quadratic irrational sides?* In other words, which right triangles have rational angles and have side lengths that each contain at most one square root? It seems natural to suspect that any such triangle is similar to a 45–45–90 or a 30–60–90 triangle. Perhaps surprisingly we prove the following.

Theorem. *The right triangles with rational angles and with rational or quadratic irrational sides are precisely the (properly scaled) 45–45–90, 30–60–90, and 15–75–90 triangles.*

Thus, there are exactly three similarity types of right triangles with rational angles and with side lengths that each contain at most one square root.

After we proved the above theorem, a search of the literature revealed a wonderful complement called the *Ailles' rectangle* named for its discoverer, Douglas Ailles [**1**]. Figure 3 shows Ailles' original rectangle which permits one to readily solve a 15–75–90 triangle. Notice that Ailles' rectangle is composed exactly of triangles satisfying the above theorem and, moreover, every similarity type appears (the alternate version of Ailles' rectangle in [**12**, pp. 87–88] does not enjoy the former property as it contains two side lengths of degree four over $\mathbb{Q}$). Thus the question mark in Figure 1 should be replaced with the 15–75–90 triangle in Ailles' rectangle!

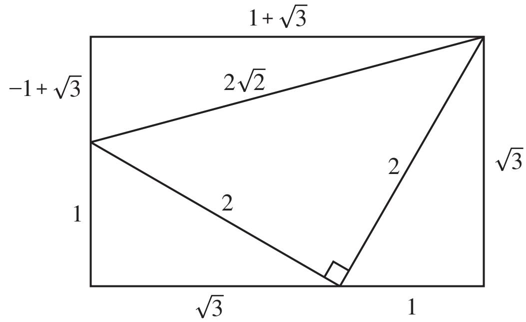

Figure 3. Original Ailles rectangle.

The above theorem and Ailles' rectangle provide a neat picture of an instance where only finitely many objects exist of some specified type. The Platonic solids, being the only convex and regular solids, are another instance. Such instances of finitely many sporadic objects can be very impressive. We hope that our theorem and Ailles' rectangle might entice some students to study field theory and number theory.

This paper is organized as follows. Section 2 reduces the proof of the main theorem to finitely many similarity types. Section 3 produces an explicit triangle, with algebraic side lengths, for each possible similarity type. Section 4 develops necessary algebraic tools. Section 5 completes the proof of the main theorem. Section 6 concludes with some remarks and questions for further study.

2. REDUCTION TO FINITELY MANY SIMILARITY TYPES. Suppose $\triangle ABC$ is a right triangle whose acute angles are rational and whose sides are each rational or quadratic irrational as in Figure 4.

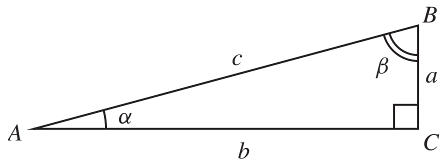

Figure 4. Right triangle $\triangle ABC$ with rational angles and rational or quadratic irrational sides.

We begin with the following observation.

Lemma 1. *Each of the numbers* $\cos\alpha$ *and* $\cos\beta$ *has degree* 1, 2, *or* 4 *over* $\mathbb{Q}$.

Proof. As $\cos\alpha = b/c \in \mathbb{Q}(b, c)$, we have the tower of fields

$$\mathbb{Q} \subseteq \mathbb{Q}(\cos\alpha) \subseteq \mathbb{Q}(b, c).$$

The degrees of these extensions satisfy

$$[\mathbb{Q}(b, c) : \mathbb{Q}] = [\mathbb{Q}(\cos\alpha) : \mathbb{Q}] \cdot [\mathbb{Q}(b, c) : \mathbb{Q}(\cos\alpha)]$$

where $[\mathbb{Q}(b, c) : \mathbb{Q}]$ equals 1, 2, or 4 since b and c each have degree 1 or 2 over $\mathbb{Q}$. This proves the result for $\cos\alpha$. The proof for $\cos\beta$ is similar. ■

Next, recall the following fact.

Lemma 2. *If* $n > 2$ *and* $\gcd(k, n) = 1$, *then*

$$\deg_{\mathbb{Q}}\left(\cos\frac{2k\pi}{n}\right) = \frac{\varphi(n)}{2}.$$

Here $\varphi(n)$ denotes the Euler totient function evaluated at n, which equals, by definition, the number of integers j such that $1 \le j \le n$ and $\gcd(j, n) = 1$. Lemma 2 is a simple consequence of the fundamental theorem of Galois theory, since if $\zeta = \cos(2k\pi/n) + i\sin(2k\pi/n)$ is a primitive nth root of unity, then $\deg_{\mathbb{Q}}\mathbb{Q}(\zeta) = \varphi(n)$ and $\mathbb{Q}(\cos(2k\pi/n)) = \mathbb{Q}(\zeta + \overline{\zeta})$ is the fixed field in $\mathbb{Q}(\zeta)$ of complex conjugation. We mention that Fact 1 follows from Lemma 2 with a bit of work.

Lemmas 1 and 2 show that we need a useful lower bound for $\varphi(n)$. A simple lower bound, sufficient for our purpose, is given below in Lemma 3 (sharper bounds are known). Recall that if $p > 1$ is prime and $a \in \mathbb{N}$, then $\varphi(p^a) = p^a - p^{a-1}$ by direct inspection. Also, φ is multiplicative: if $\gcd(m, n) = 1$, then $\varphi(mn) = \varphi(m)\varphi(n)$ (see [**5**, p. 15]).

Lemma 3. $\varphi(n) \geq \sqrt{n/2}$.

Proof. The result is clear for $n = 1$, so let $n = 2^a p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be a prime factorization of n where $a \geq 0$, the p_js are distinct positive odd primes, and $a_j \geq 1$ for each j. Now, for each j we have

$$\varphi(p_j^{a_j}) = p_j^{a_j} - p_j^{a_j-1} = p_j^{a_j-1}(p_j - 1) \geq p_j^{a_j-1} p_j^{1/2} = p_j^{a_j-1/2} \geq p_j^{a_j/2},$$

where the first inequality used $p_j \geq 3$. For the prime 2, we have $\varphi(2^0) = 1$ and if $a \geq 1$, then

$$\varphi(2^a) = 2^a - 2^{a-1} = 2^{a-1}(2-1) = 2^{a-1}.$$

In either case, we have $\varphi(2^a) \geq (2^{a/2})/\sqrt{2}$. Using multiplicativity of φ, we obtain

$$\varphi(n) = \varphi(2^a)\varphi(p_1^{a_1})\varphi(p_2^{a_2}) \cdots \varphi(p_k^{a_k}) \geq \frac{2^{a/2}}{\sqrt{2}} p_1^{a_1/2} p_2^{a_2/2} \cdots p_k^{a_k/2} = \frac{\sqrt{n}}{\sqrt{2}}. \qquad \blacksquare$$

Combining the above, we obtain the following.

Lemma 4. *The radian measures α and β both lie in the set*

$$S = \left\{\frac{\pi}{15}, \frac{\pi}{12}, \frac{\pi}{10}, \frac{\pi}{8}, \frac{2\pi}{15}, \frac{\pi}{6}, \frac{\pi}{5}, \frac{\pi}{4}, \frac{4\pi}{15}, \frac{3\pi}{10}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{2\pi}{5}, \frac{5\pi}{12}, \frac{7\pi}{15}\right\}.$$

Proof. By Lemma 1, $\cos\alpha$ has degree 1, 2, or 4 over $\mathbb{Q}$. Let $\alpha = 2k\pi/n$ where $\gcd(k, n) = 1$, $k \in \mathbb{N}$, and $n > 2$ (since $\alpha < \pi/2$). By Lemmas 2 and 3, we need only consider the cases $3 \leq n \leq 128$. Either by hand or, better, using a computational algebra system such as MAGMA, we compute $\varphi(n)/2$ for these values of n and find that n lies in the set

$$\{3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30\}.$$

For each of these values of n, one simply produces the corresponding values of k with $\gcd(k, n) = 1$ and $0 < \alpha = 2k\pi/n < \pi/2$. The proof for β is identical. $\blacksquare$

As α and β are complementary and lie in S, we obtain our desired reduction to a finite set of possible similarity types.

Proposition 1. *The multiset $\{\alpha, \beta\}$ lies in the set*

$$T = \left\{\left\{\frac{\pi}{6}, \frac{\pi}{3}\right\}, \left\{\frac{\pi}{4}, \frac{\pi}{4}\right\}, \left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}, \left\{\frac{\pi}{10}, \frac{2\pi}{5}\right\}, \left\{\frac{\pi}{8}, \frac{3\pi}{8}\right\}, \left\{\frac{\pi}{5}, \frac{3\pi}{10}\right\}\right\}.$$

We mention that the previous result may be obtained using sine instead of cosine. Using tangent, however, yields 12 possibilities instead of six and leaves one with more work. The first two elements of T yield the two well-known similarity types in the main theorem. The remainder of the proof is concerned with the last four elements of T.

3. EXPLICIT TRIANGLES. In this section, we produce four explicit right triangles with algebraic side lengths, each triangle representing one of the last four similarity types in T (see Proposition 1 above). The lengths of the legs in these triangles will be obtained using the tangent analogues of the Chebyshev polynomials.

For each $m \in \mathbb{N}$, define

$$F_m(x) = \tan(m \arctan x).$$

These are the tangent analogues of the Chebyshev polynomials of the first kind for cosine. Let $\theta = \arctan x$; then De Moivre's formula yields

$$\begin{aligned} F_m(x) = \frac{\sin m\theta}{\cos m\theta} &= \frac{1}{i}\frac{(\cos\theta + i\sin\theta)^m - (\cos\theta - i\sin\theta)^m}{(\cos\theta + i\sin\theta)^m + (\cos\theta - i\sin\theta)^m} \\ &= \frac{1}{i}\frac{(1+ix)^m - (1-ix)^m}{(1+ix)^m + (1-ix)^m} = \frac{\operatorname{Im}(1+ix)^m}{\operatorname{Re}(1+ix)^m} = \frac{p_m(x)}{q_m(x)}, \end{aligned}$$

where the last equality defines the polynomials $p_m(x)$ and $q_m(x)$ in $\mathbb{Z}[x]$. Thus, each $F_m(x)$ is a rational function with integer coefficients.

Notice that $p_m(x)$ equals the alternating sum of the odd power terms in the binomial expansion $(1+x)^m$, the first few being

$$p_1(x) = x, \quad p_2(x) = 2x, \quad p_3(x) = 3x - x^3, \quad p_4(x) = 4x - 4x^3.$$

Plainly, if $\tan(k\pi/n)$ exists (i.e., $k \not\equiv n/2 \mod n$), then $\tan(k\pi/n)$ is a root of $F_n(x)$ and of $p_n(x)$. In other words, the minimal polynomial of $\tan(k\pi/n)$ may be obtained by factoring $p_n(x)$ in $\mathbb{Z}[x]$ using MAGMA and then choosing the correct irreducible factor (see [**4**] for another rigorous approach).

Here is an example. Let $n = 10$. Then

$$p_{10}(x) = 10x^9 - 120x^7 + 252x^5 - 120x^3 + 10x$$

factors in $\mathbb{Z}[x]$ into

$$p_{10}(x) = 2x(x^4 - 10x^2 + 5)(5x^4 - 10x^2 + 1).$$

Calculation shows that $\tan(\pi/10) \neq 0$ is not a root of $x^4 - 10x^2 + 5$; hence it must be a root of $\psi(x) = 5x^4 - 10x^2 + 1$ (this is a rigorous argument!). Therefore

$$\tan\left(\frac{\pi}{10}\right) = \sqrt{\frac{5 - 2\sqrt{5}}{5}}.$$

Recalling Figure 4, we may set $a = \sqrt{5 - 2\sqrt{5}}$, $b = \sqrt{5}$, and, by the Pythagorean theorem, $c = \sqrt{10 - 2\sqrt{5}}$.

Repeating this process for $\pi/12$, $\pi/8$, and $\pi/5$, we obtain the four triangles I–IV described in Table 2.

Table 2. Data for four right triangles I–IV, namely the radian measure α of an acute angle, the minimal polynomial $\psi(x)$ of $\tan\alpha$ over $\mathbb{Q}$, $\tan\alpha$ in radical form, and the side lengths a, b, and c as in Figure 4 above.

$\triangle$	α	$\psi(x)$	$\tan(\alpha)$	a	b	c
I	$\pi/12$	x^2-4x+1	$2-\sqrt{3}$	$2-\sqrt{3}$	1	$\sqrt{8-4\sqrt{3}}$
II	$\pi/10$	$5x^4-10x^2+1$	$\sqrt{(5-2\sqrt{5})/5}$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{5}$	$\sqrt{10-2\sqrt{5}}$
III	$\pi/8$	x^2+2x-1	$\sqrt{2}-1$	$\sqrt{2}-1$	1	$\sqrt{4-2\sqrt{2}}$
IV	$\pi/5$	x^4-10x^2+5	$\sqrt{5-2\sqrt{5}}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{6-2\sqrt{5}}$

Each triangle I–IV appears to contain at least one side whose length has degree 4 over $\mathbb{Q}$. However, looks can be deceiving when it comes to radical expressions, as is well known to anyone who has played with the cubic formula. In the present situation, we have

$$\sqrt{6-2\sqrt{5}} = \sqrt{\left(\sqrt{5}-1\right)^2} = \sqrt{5}-1.$$

This observation indicates that we will need tools to recognize squares and nonsquares among quadratic irrationals. These tools are developed in the next section.

4. ALGEBRAIC TOOLS. Recall that a *number field* is a subfield of $\mathbb{C}$ whose dimension as a vector space over $\mathbb{Q}$ is finite. Being a subfield of $\mathbb{C}$, each number field is an integral domain. A *quadratic number field* K is a number field with $[K:\mathbb{Q}]=2$. It follows that $K=\mathbb{Q}(\sqrt{d})$ for some squarefree integer $d\notin\{0,1\}$. Plainly

$$\mathbb{Q}(\sqrt{d}) = \mathbb{Q}[\sqrt{d}] = \{r+s\sqrt{d} \mid r,s\in\mathbb{Q}\}$$

(the first equality is a special case of a more general fact [**11**, §1.3]).

If K is a number field, then the *ring of integers of* K is by definition

$$\mathcal{O}_K = \{\gamma\in K \mid f(\gamma)=0 \text{ for some monic } f\in\mathbb{Z}[x]\}.$$

Historically, it was nontrivial to arrive at this correct definition of $\mathcal{O}_K$ (see [**8**, p. 38]). Indeed, $\mathcal{O}_K$ is a ring and $\mathbb{Z}\subseteq\mathcal{O}_K\subset K$ (see [**11**, §2.3]). For example, $\mathcal{O}_{\mathbb{Q}}=\mathbb{Z}$ (the so-called *rational integers*) and $\mathcal{O}_{\mathbb{Q}(i)}=\mathbb{Z}[i]$ (the so-called *Gaussian integers*). Direct verification [**11**, §3.1] shows that if $K=\mathbb{Q}(\sqrt{d})$ is a quadratic number field where $d\notin\{0,1\}$ is a squarefree rational integer, then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if} \quad d\not\equiv 1 \mod 4, \\ \mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{d}] & \text{if} \quad d\equiv 1 \mod 4, \end{cases}$$

where

$$\mathbb{Z}[\sqrt{d}] = \{a+b\sqrt{d} \mid a,b\in\mathbb{Z}\} \quad \text{and}$$

$$\mathbb{Z}\left[\frac{1}{2}+\frac{1}{2}\sqrt{d}\right] = \left\{a+\frac{b}{2}+\frac{b}{2}\sqrt{d} \;\middle|\; a,b\in\mathbb{Z}\right\}.$$

The norm function N is an important tool for the study of quadratic number fields as we now recall. Fix a squarefree integer $d\notin\{0,1\}$ and let $K=\mathbb{Q}(\sqrt{d})$. Define *conjugation* on K, in analogy with complex conjugation, as follows. If $\mu = r + s\sqrt{d} \in K$ where $r, s \in \mathbb{Q}$, then

$$\overline{\mu} = r - s\sqrt{d} \in K.$$

This conjugation is well defined (since each element of K is uniquely expressible as $r + s\sqrt{d}$ for $r, s \in \mathbb{Q}$) and is multiplicative:

$$\overline{\mu\nu} = \overline{\mu}\,\overline{\nu} \quad \text{for every} \quad \mu, \nu \in K.$$

Now, if $\mu = r + s\sqrt{d} \in K$, where $r, s \in \mathbb{Q}$, then define the *norm* of μ by

$$N(\mu) = \mu\overline{\mu} = r^2 - s^2 d \in \mathbb{Q}.$$

Multiplicativity of conjugation and commutativity of multiplication on K imply that the norm is multiplicative:

$$N(\mu\nu) = N(\mu)N(\nu) \quad \text{for every} \quad \mu, \nu \in K.$$

In particular, the restriction of N to $\mathcal{O}_K$ is multiplicative and, furthermore, takes rational integer values:

$$N : \mathcal{O}_K \to \mathbb{Z}.$$

The latter fact is obvious if $d \not\equiv 1 \mod 4$ and is easily verified directly when $d \equiv 1 \mod 4$. Thus, we obtain the following sufficient condition for the recognition of nonsquares in $\mathcal{O}_K$.

Proposition 2. *Let K be a quadratic number field and let $\mu \in \mathcal{O}_K$. If $N(\mu)$ is not a square in $\mathbb{Z}$, then μ is not a square in $\mathcal{O}_K$.*

The converse of Proposition 2 does not hold in general, as shown by simple examples such as $-4 + 2\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$.

Next, we develop a sufficient condition for the recognition of squares in $\mathcal{O}_K$.

Question. *Let R be a commutative ring. If $\alpha, \beta, \gamma \in R - \{0_R\}$ and*

$$\alpha \cdot \beta^2 = \gamma^2,$$

then is α necessarily a square in R?

A straightforward exercise (left to the reader) shows that the question has an affirmative answer for each unique factorization domain R, in particular for the rational integers $\mathbb{Z}$. This property of $\mathbb{Z}$ is at the heart of the age-old question: *does $\sqrt{2}$ lie in $\mathbb{Q}$?* If so, then $2q^2 = p^2$ for some nonzero rational integers p and q. The property now implies that 2 is a square in $\mathbb{Z}$, a contradiction. A similar argument shows that only the square rational integers have square roots lying in $\mathbb{Q}$.

The aforementioned property of the positive integers $\mathbb{Z}^+$ was known to Euclid. Proposition 22 in book VIII of Euclid's *Elements* [**7**, p. 379] states that *if three numbers be in continued proportion, and the first be a square, the third will also be a square.* That is, if $a, b, c \in \mathbb{Z}^+$, $a : b : c$, and $a = d^2$ for some $d \in \mathbb{Z}^+$, then $c = e^2$ for some $e \in \mathbb{Z}^+$. The continued proportion $a : b : c$ means $a/b = b/c$ or $ca = b^2$. Therefore, Euclid's Proposition 22 is equivalent to: if $c, d, b \in \mathbb{Z}^+$ and $cd^2 = b^2$, then $c = e^2$ for some $e \in \mathbb{Z}^+$.

The above question has a negative answer for many rings. If $R = \mathbb{Z}/6\mathbb{Z}$, then $5 \cdot 3^2 = 3^2$, but 5 is not a square in R. For present purposes, a more relevant counterexample is the integral domain $R = \mathbb{Z}[2i]$; in R we have $-1 \cdot 2^2 = (2i)^2$, but -1 is not a square in R. The problem is that $\mathbb{Z}[2i]$ is not integrally closed.

Recall that a ring R is *integrally closed* if each element of its field of fractions

$$F = \{\alpha/\beta \mid \alpha, \beta \in R \text{ and } \beta \neq 0_R\}$$

that is integral over R (meaning the element is a root of a monic polynomial in $R[x]$) actually lies in R. It is not difficult to see that $\mathbb{Q}[i]$ is the field of fractions of $\mathbb{Z}[2i]$ and $i \in \mathbb{Q}[i]$ is integral over $\mathbb{Z}[2i]$, but $i \notin \mathbb{Z}[2i]$. In fact, $\mathcal{O}_{\mathbb{Q}[i]} = \mathbb{Z}[i]$ and so in a sense $\mathbb{Z}[2i]$ is missing some integers, namely i. The next lemma (see [**11**, p. 106] for a proof) states that this problem does not arise in rings of present interest.

Lemma 5. *If K is a number field, then $\mathcal{O}_K$ is integrally closed.*

This yields the following sufficient condition for the recognition of squares in $\mathcal{O}_K$.

Proposition 3. *Let K be a number field and $R = \mathcal{O}_K$. If $\alpha, \beta, \gamma \in R - \{0_R\}$ and $\alpha \cdot \beta^2 = \gamma^2$, then α is a square in R.*

Proof. If $\alpha, \beta, \gamma \in R - \{0_R\}$ and $\alpha \cdot \beta^2 = \gamma^2$, then $x^2 - \alpha \in R[x]$ has γ/β as a root. The previous lemma implies $\gamma/\beta \in R$ and, thus, $\alpha = (\gamma/\beta)^2$ is a square in R. ∎

We will make use of the following technical result.

Lemma 6. *Let $\alpha = \sqrt{x + y\sqrt{z}}$, where $x, y, z \in \mathbb{Z}$, $y \neq 0$, and $z > 1$ is squarefree. Then $\deg_{\mathbb{Q}} \alpha = 2$ or $\deg_{\mathbb{Q}} \alpha = 4$. Furthermore, $\deg_{\mathbb{Q}} \alpha = 2$ if and only if $x + y\sqrt{z} = \left(a + b\sqrt{z}\right)^2$ for some nonzero $a, b \in \mathbb{Z}$.*

Proof. Notice that $\alpha^2 = x + y\sqrt{z} \notin \mathbb{Q}$ is a root of $t^2 - (2x)t + (x^2 - y^2 z) \in \mathbb{Z}[t]$, and so $\deg_{\mathbb{Q}} \alpha^2 = 2$. The tower of fields

$$\mathbb{Q} \overset{2}{\subset} \mathbb{Q}(\alpha^2) \overset{d}{\subseteq} \mathbb{Q}(\alpha)$$

has $d = 1$ or $d = 2$ since α is a root of $t^2 - \alpha^2 \in \mathbb{Q}(\alpha^2)[t]$. As $\deg_{\mathbb{Q}} \alpha = 2d$, the first part of the lemma follows. Next

$$\begin{aligned} \deg_{\mathbb{Q}} \alpha = 2 &\iff d = 1 \\ &\iff \alpha \in \mathbb{Q}\left(\alpha^2\right) \\ &\iff \alpha = \frac{a + b\sqrt{z}}{c} \end{aligned}$$

for some $a, b, c \in \mathbb{Z}$ with $\gcd(a, b, c) = 1$ and $c > 0$. In this case

$$x + y\sqrt{z} = \alpha^2 = \left(\frac{a + b\sqrt{z}}{c}\right)^2 = \frac{a^2 + b^2 z}{c^2} + \frac{2ab}{c^2}\sqrt{z}$$

and so

$$x = \frac{a^2 + b^2 z}{c^2} \quad \text{and} \quad y = \frac{2ab}{c^2}$$

since $\{1, \sqrt{z}\}$ is a linearly independent set over $\mathbb{Q}$. Hence $c^2 \mid a^2 + b^2 z$ and $c^2 \mid 2ab$. If p is a rational prime dividing c, then $c^2 \mid 2ab$ implies $p \mid a$ or $p \mid b$, and $c^2 \mid a^2 + b^2 z$ implies $p \mid a$ and $p \mid b$ since z is squarefree. This contradicts $\gcd(a, b, c) = 1$, so $c = 1$. Finally, $a, b \neq 0$ since otherwise $\deg_{\mathbb{Q}} \alpha^2 = 1$, a contradiction. ■

We recall one last technical tool (see [**9**] for a proof) before we proceed to the proof proper of the main theorem.

Lemma 7. *Let $d_1, d_2, \ldots, d_n$ be distinct squarefree rational integers greater than one. Then $D = \{1, \sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\}$ is a linearly independent set over $\mathbb{Q}$.*

5. COMPLETION OF THE PROOF OF THE THEOREM. In this section, we determine whether there exist triangles of the last four similarity types in T (see Proposition 1 above) and with rational or quadratic irrational sides. We begin with the last similarity type 36–54–90, which is represented by triangle IV (see Table 2 above) with side lengths

$$\begin{aligned} a &= \sqrt{5 - 2\sqrt{5}}, \\ b &= 1, \\ c &= \sqrt{6 - 2\sqrt{5}} = \sqrt{5} - 1. \end{aligned}$$

Let $K = \mathbb{Q}\left(\sqrt{5}\right)$ and recall from the previous section that

$$\mathcal{O}_K = \mathbb{Z}\left[\frac{1}{2} + \frac{1}{2}\sqrt{5}\right] \supset \mathbb{Z}[\sqrt{5}].$$

Notice that $N(5 - 2\sqrt{5}) = 5$ is not a square in $\mathbb{Z}$, and so $5 - 2\sqrt{5}$ is not a square in $\mathcal{O}_K$ by Proposition 2 above (hence, $5 - 2\sqrt{5}$ is not a square in $\mathbb{Z}[\sqrt{5}] \subset \mathcal{O}_K$). Therefore, $\deg_{\mathbb{Q}} a = 4$ by Lemma 6 and triangle IV is ruled out.

It remains to rule out all triangles similar to triangle IV. Suppose, by way of contradiction, that a triangle, called IV′, is similar to triangle IV and satisfies the conditions in the theorem. Thus, the side lengths of IV′ are $\lambda\sqrt{5 - 2\sqrt{5}}$, $\lambda \cdot 1$, and $\lambda(\sqrt{5} - 1)$ for some $\lambda > 0$. Therefore, λ itself is rational or quadratic irrational. Scaling further by a positive rational integer, we may assume that

$$\lambda = x + y\sqrt{d} > 0, \tag{1}$$

$$\lambda(\sqrt{5} - 1) = e + f\sqrt{d_1} > 0, \quad \text{and} \tag{2}$$

$$\lambda\sqrt{5 - 2\sqrt{5}} = g + h\sqrt{d_2} > 0, \tag{3}$$

where all variables (except possibly λ) are rational integers and $d, d_1, d_2 > 1$ are all squarefree. By (1), x and y are not both zero. Further, $y \neq 0$ since otherwise $\lambda\sqrt{5 - 2\sqrt{5}}$ has degree 4 over $\mathbb{Q}$ instead of the required degree 1 or 2.

The idea of the remainder of the proof is quite simple. We suspect that equations (1)–(3) imply that $d = d_2 = 5$. But then squaring equation (3) yields

$$\left(x + y\sqrt{5}\right)^2\left(5 - 2\sqrt{5}\right) = \left(g + h\sqrt{5}\right)^2.$$

By Proposition 3, the previous equation implies that $5 - 2\sqrt{5}$ is a square in $\mathcal{O}_K$, which we have already seen is false. Thus, triangle IV$'$ does not exist. It remains to show that $d = d_2 = 5$.

Claim 1. $d = 5$.

Proof. Multiplying out equation (2) yields

$$-x + x\sqrt{5} + y\sqrt{5d} - y\sqrt{d} = e + f\sqrt{d_1}. \tag{4}$$

As $y \neq 0$, if $5 \nmid d$, then equation (4) contradicts Lemma 7 on the linear independence of square roots. Therefore, $d = 5d_0$ where $5 \nmid d_0$ (since d is squarefree) and

$$-x + x\sqrt{5} + 5y\sqrt{d_0} - y\sqrt{5d_0} = e + f\sqrt{d_1}. \tag{5}$$

If $d_0 > 1$, then equation (5) contradicts Lemma 7. Hence, $d_0 = 1$ and $d = 5$. ■

Claim 2. $x \neq 0$.

Proof. Otherwise $y > 0$ (since $\lambda > 0$) and

$$g + h\sqrt{d_2} = y\sqrt{5}\sqrt{5 - 2\sqrt{5}} = \sqrt{25y^2 - 10y^2\sqrt{5}}.$$

As $g + h\sqrt{d_2}$ has degree 1 or 2 over $\mathbb{Q}$, Lemma 6 implies that $25y^2 - 10y^2\sqrt{5}$ is a square in $\mathbb{Z}\left[\sqrt{5}\right]$. However, the norm of this element is 5^3y^4 which is not a square in $\mathbb{Z}$, contradicting Proposition 2. ■

Claim 3. $d_2 = 5$.

Proof. Otherwise, square both sides of (3) and conclude, by Lemma 7, that the coefficient $-2(x^2 + 5y^2) + 10xy$ of $\sqrt{5}$ must equal zero. Setting this coefficient equal to zero and solving the resulting quadratic in x we obtain

$$x = \frac{5y \pm y\sqrt{5}}{2}.$$

This is a contradiction since $x \in \mathbb{Z}$ and $y \neq 0$. ■

This completes the proof that no triangle similar to triangle IV has rational or quadratic irrational sides. A virtually identical argument applies to triangles II and III (for triangle II, start with the similar triangle obtained by scaling triangle II by $\sqrt{5}$). We leave the details to the reader.

In the case of triangle I, the above argument begins to break down when one attempts to prove $x \neq 0$. Assuming $x = 0$, one concludes that, by Lemma 6, $3y^2\left(8 - 4\sqrt{3}\right)$ is a square in $\mathbb{Z}\left[\sqrt{3}\right]$. The analogous statements for triangles II–IV were contradictions as shown by the norm. However, here the norm of the potential square equals $2^4 3^2 y^4$ which is a square in $\mathbb{Z}$. This is not fatal and one can prove by elementary means that $x \neq 0$. One then has $\lambda = x + y\sqrt{3}$ where x and y are nonzero rational integers. The attempt to prove $d_2 = 3$, however, is fatal. In an attempted proof by contradiction, one assumes that $d_2 \neq 3$ and solves the quadratic in x that is the coefficient of $\sqrt{3}$ (as above) and finds that $x = y$ or $x = 3y$. Examining the case $x = y$, one finds that $\lambda = 1 + 1\sqrt{3}$ scales triangle I to a triangle with side lengths $a' = \sqrt{3} - 1$, $b' = 1 + \sqrt{3}$, and $c' = 2\sqrt{2}$. This yields the third similarity type in the main theorem.

6. CONCLUDING REMARKS. Having proved the theorem, it is straightforward to find all triangles satisfying the conditions in the theorem. Making use of Lemma 7, the reader may verify the following. Each 45–45–90 triangle satisfying the conditions in the theorem is obtained from the standard one with opposite side lengths $\left(1, 1, \sqrt{2}\right)$ by: (i) scaling by $\sqrt{d}$, where $d \geq 1$ is squarefree, and then scaling by an arbitrary element in $\mathbb{Q}^+$, or (ii) scaling by $x + y\sqrt{2} > 0$, where $x, y \neq 0$ and $\gcd(x, y) = 1$, and then scaling by an arbitrary element in $\mathbb{Q}^+$. The same result holds for each 30–60–90 triangle similar to the one with opposite side lengths $\left(1, \sqrt{3}, 2\right)$ except that in (ii) one must use $x + y\sqrt{3} > 0$. Identifying triangles that differ by a scale factor in $\mathbb{Q}^+$, we see that 45–45–90 and 30–60–90 triangles each have a countably infinite number of equivalence classes of triangles satisfying the conditions in the theorem.

In contrast, one may verify that 15–75–90 triangles have exactly two such equivalence classes with representatives the "standard" one with opposite side lengths $\left(\sqrt{3} - 1, 1 + \sqrt{3}, 2\sqrt{2}\right)$ and the standard one scaled by $\sqrt{3}$.

The theorem above suggests many natural questions and we mention two. First, what happens when one drops the requirement in the theorem that the triangles contain a right angle? Using the law of cosines, Lemmas 2 and 3, and a computer algebra system (such as MAGMA), the problem reduces to checking 200 similarity types. Second, we focused on side lengths of degree at most 2 in relation to grade school triangles. The argument above showing that there exist only a finite number of possible similarity types also applies to right triangles with rational angles and side lengths of degree at most d. Actually determining which of the similarity types work appears to be much more difficult for general $d > 2$.

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Robert Burton (1577–1640) on the Diversion of Melancholy

"I would for these causes wish him that is melancholy, to use both humane and divine authors, voluntarily to impose some taske upon himselfe, to divert his melancholy thoughts: To study the art of memory, *Cosmus Rosselius, Pet. Ravennas, Scenkelius detectus, or practice Brachygraphy, &*. that will aske a great deale of attention: or let him demonstrate a proposition in *Euclide* in his five last bookes, extract a square root, or study *Algebra*. Then which as *Clavius* holds, *in all humane disciplines nothing can be more excellent and pleasant, so abstruse and recondite, so bewitching, so miraculous, so ravishing, so easie withall and full of delight, omnem humanum captum superare videtur*."

Robert Burton, *The Anatomy of Melancholy*, vol. 2,
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—Submitted by Robert Haas, Cleveland Heights, OH