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Open Books

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Let W^{n-1} be a compact manifold with boundary and and let $h\colon W \to W$ be a homeomorphism which is the identity on ∂W.

In $W \times [0,1]$ identify $(x,1)$ to $(h(x),0)$ for $x \in W$ and identify (x,t) to $(x,0)$ for $x \in \partial W, t \in [0,1]$.

The resulting space is a closed n-manifold $M^n = M(h)$ called an *open book* with *monodromy* h. If $\Psi\colon W \times [0,1] \to M^n$ is the identification map then $\Psi(\partial W \times \{0\})$ is the *binding* and $\Psi(W \times \{t\}), t \in [0,1]$ is a *page*.

If W is oriented, then $W \times [0,1]$ is canonically oriented and $M(h)$ is oriented in such a way that Ψ has degree 1.

Example 1. If $h\colon S^1 \times [0,1] \to S^1 \times [0,1]$ is defined by $h(z,t) = (e^{2\pi i t} \cdot z, t)$ then $M(h) \approx S^3$.

Theorem 1 (Winkelnkemper). *If M^n is 1-connected and $n > 5$, then M^n is an open book iff index $M^n = 0$. (index $M^n = 0$ by definition if n is not a multiple of 4).*

Theorem 2 (Tamura($k > 2$), A' Campo ($k = 2$)). *If M^{2k+1} is $(2k-1)$-connected and $k \geq 2$, then M is an open book with S^{2k-1} as binding.*

Theorem 3 ((Alexander)1923). *Any orientable closed M^3 is an open book.*

If M^3 can be exhibited as an open book where the page has genus g and d boundary components we say that M^3 is a book ot type (g, d). Our main result is:

Theorem 4. *Let M^3 be an orientable closed 3-manifold. Then, for all but a finite number of pairs (g, d) with $g \geq 0, d > 0, M^3$ is a book of type (g, d). (In particular M^3 has a fibered link with a prescribed number of components and a fibered link of genus 0).*

Corollary 1. *Let M^3 be orientable closed connected and d a positive integer. Then there exists a codimension 1 foliation of M with precisely d compact leaves.*

Plumbing Lemma. *Let $W^{n-1} = W_0^{n-1} \cup W_1^{n-1}$ where W, W_0 and W_1 are manifolds with boundary. Suppose $W_0 \cap W_1 \approx D^{n-1}$ and that $W_i \cap \partial W_j$ is bicollared in W_i if $i \neq j$. Let $h_i \colon W \to W$ be a homeomorphism such that $h_i \mid \overline{W - W_i}$ is the identity $(i = 0, 1)$. Then $M(h_0 \circ h_1) \approx M(h_0 \mid W_0) \# M(h_1 \mid W_1)$.*

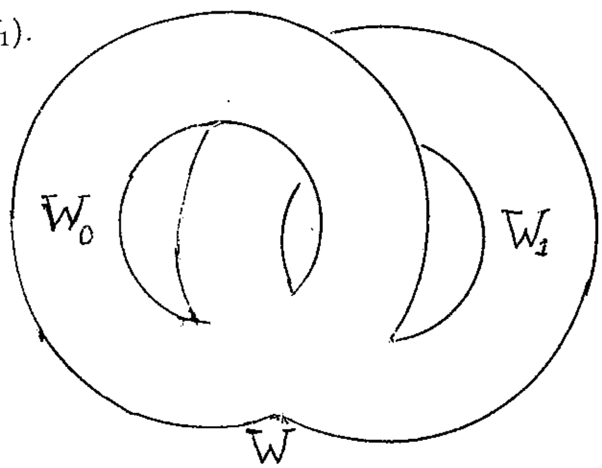


Fig. 1

Proof: For $i = 0, 1$ and $x \in \partial W_i \cap \operatorname{int} W_{1-i}$ identify in $M(h_i \mid W_i)$ and in $M(h_0 \circ h_1)$ the image under the identification map of $\{x\} \times [\frac{i}{2}, \frac{i+1}{2}]$ to a point.

The quotient space M of $M(h_0 \circ h_1)$ is homeomorphic to $M(h_0 \circ h_1)$: a homeomorphism is defined by sending (x, t) in M to $\left(x, \frac{d(x, W_0 \cap \partial W_1)\alpha(t) + d(x, W_1 \cap \partial W_0)(1-\alpha(1-t))}{d(x, W_0 \cap \partial W_1) + d(x, W_1 \cap \partial W_0)}\right)$ in $M(h_0 \circ h_1)$ where d is a distance on W, α is a map from $[0, 1]$ to $[0, 1]$ linear on the subintervals $[0, \frac{1}{2}], [\frac{1}{2}, 1]$ and $\alpha(0) = \alpha(\frac{1}{2}) = 0, \alpha(1) = 1$.

Similarly the quotient space M_i of $M(h_i \mid W_i)$ is homeomorphic to $M(h_i \mid W_i)$.

Let $N_i = \Psi(W_i \times [0, 1] - (W_i \cap \operatorname{int} W_{1-i}) \times (\frac{1-i}{2}, \frac{2-i}{2}))$ where $\Psi \colon W \times [0, 1] \to M$ is the identification map. (See the schematic Figure 2)

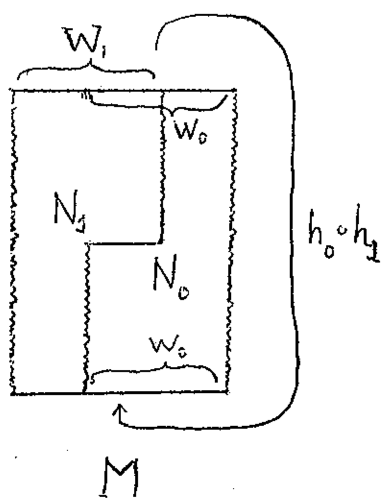


Fig. 2

Since $h_0 h_1 = h_0$ on $W_0 - \operatorname{int} W_1$ one sees that N_0 is homeomorphic to the complement, in M_0, of the interior of the n-disk corresponding to $(W_0 \cap W_1) \times [\frac{1}{2}, 1]$.

Also, the complement in M_1 of the interior of the n-disk which corresponds to $(W_0 \cap W_1) \times \left[0, \frac{1}{2}\right]$ is homeomorphic to N_1. The homeomorphism is defined by mapping the point of M_1 which corresponds to (x,t), with $(x,t) \notin \mathrm{int}(W_0 \cap W_1) \times (0, \frac{1}{2})$, to $\Psi(h_0(x), t)$ if $t < \frac{1}{2}$ and to $\Psi(x,t)$ if $t \geq \frac{1}{2}$. ∎

Now, suppose M_0^3 is a book of type $(0,d), d > 1$, page W_0 and monodromy h_0. Take $M_1^3 = S^3$ with page an annulus W_1 and monodromy h_1. (see Example 1). Let $W = W_0 \cup W_1$ as shown in Figure 3.

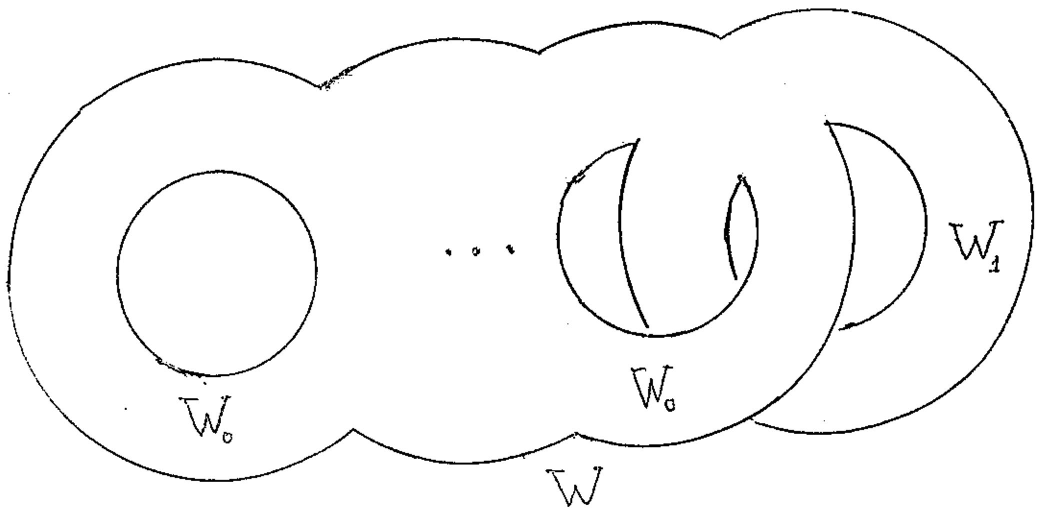


Fig. 3

W has genus 1 and $d-1$ boundary components.

Extend h_i to a homeomorphism of W onto W in such a way that $h_i \mid W - W_i =$ identity.

Then by the plumbing lemma $M(h_0 \circ h_1) \approx M_0^3 \# M_1^3 \approx M_0^3$ so that M_0^3 is a book of type $(1, d-1)$.

Iterating this construction we get that, for $0 \leq i < d, M_0^3$ is a book of type $(i, d-i)$. If, instead, we construct W as shown in Figure 4 and apply the plumbing lemma we obtain that M_0^3 is a book of type $(0, d+1)$ and, iterating, that M_0^3 is a book of type $(0, d')$ for any $d' \geq d$.

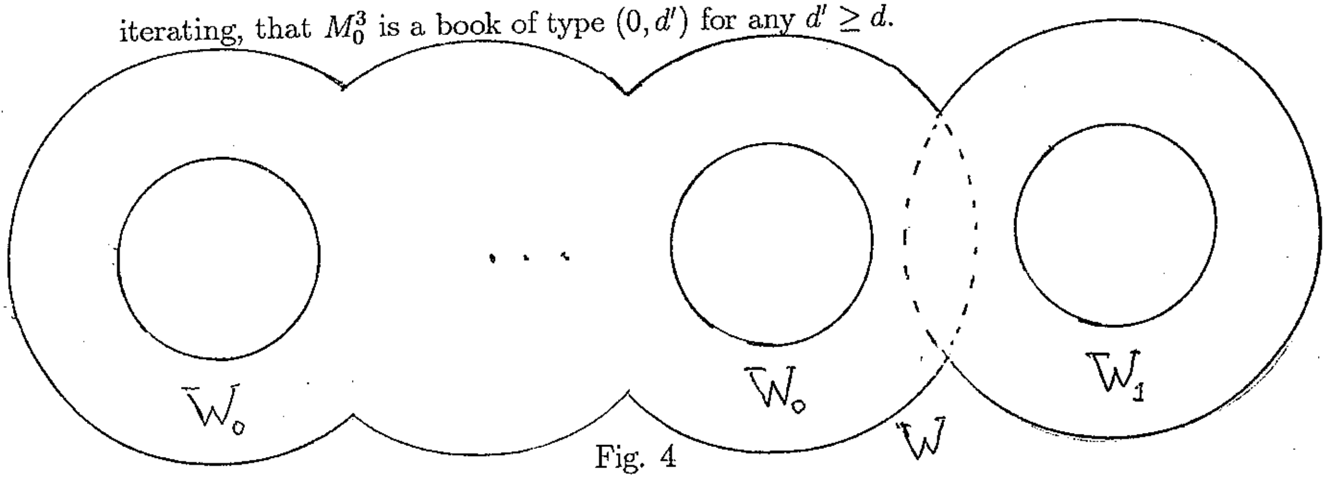


Fig. 4

These observations show that if M^3 is a book of type $(0, d)$ and $g + d' \geq d$ where $g \geq 0, d' > 0$, then M^3 is also a book of type (g, d').

Thus, to prove the theorem it suffices to show that every closed orientable 3-manifold is a book of type $(0, d)$ for some d. But this can be proved using Lickorish's work in "Foliations of 3-manifolds" as follows.

Let $S^3 = \bigcup_{z \in S^1} W_z, W_z \approx D^2$ be the standard open book decomposition of S^3:

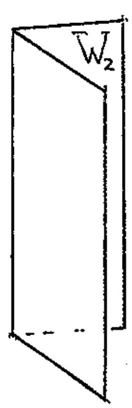


Fig. 5

Let $\varphi_i\colon S^1 \to S^3 (i = 1, \dots, n)$ be disjoint embeddings such that $\varphi_i(z) \in \operatorname{int} W_z \,\forall i \,\forall z \in S^1$. The link $\overset{n}{\underset{i=1}{\cup}} \varphi_i(S^1)$ is called a pure link (see Birman's book).

Lickorish proves: Every orientable closed M^3 can be obtained by Milnor-Wallace surgery on a pure link. That is,

$M^3 = (S^3 - \overset{n}{\underset{i=1}{\cup}} \varphi_i(S^1 \times D^2)) \cup (D^2 \times S^1)_1 \cup \dots \cup (D^2 \times S^1)_n$ with $\varphi_i(u, v) \sim (u, v) \in (\partial D^2 \times S^1)_i$.

Here φ_i has been extended to a homeomorphim of $S^1 \times D^2$ into S^3 in such a way that $\varphi_i(\{z\} \times D^2) \subset \operatorname{int} W_z$.

Then, if $W'_z = W_z - \overset{n}{\underset{i=1}{\cup}} \varphi_i(\{z\} \times \operatorname{int} D^2)) \cup (\overset{n}{\underset{i=1}{\cup}}([0, z] \times S^1)_i)$, where $[0, z]$ is the segment from $0 \in D^2$ to $z \in \partial D^2 = S^1, \{W'_z\}$ are the (planar) pages of a book decomposition of M^3.

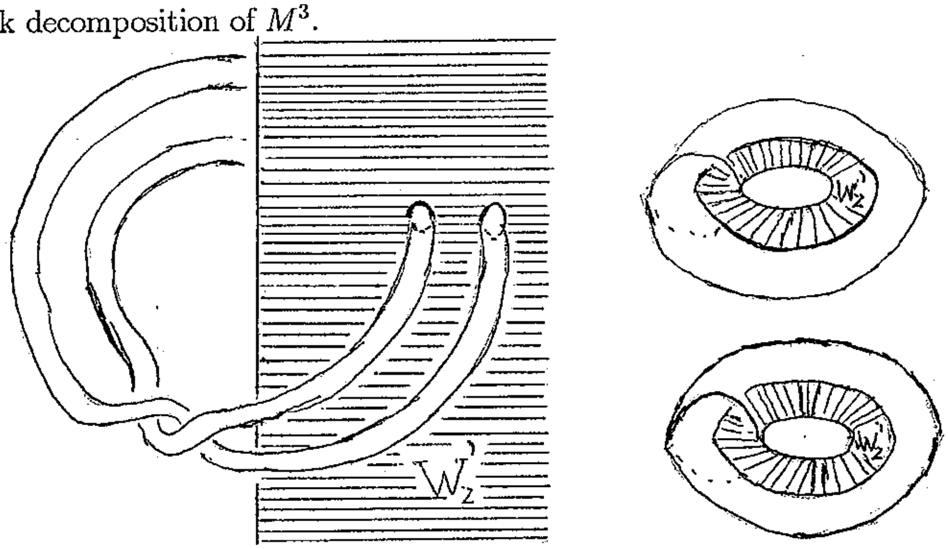


Fig. 6

The fact that any orientable closed 3-manifold M^3 has a fibered knot, *i.e* is a book of type $(g,1)$ for some g, can also be proved "a la Alexander" as follows.

By Alexander(details can be found in Montesinos' "Una nota a un teorema de Alexander") $M^3 \approx M(l,\omega)$ where $M(l,\omega)$ is the covering of S^3 branched over a link l associated to a transitive representation $\omega\colon \pi_1(S^3 - l, \infty) \to S_q$ sending meridians to transpositions.

l is pictured as a closed r-braid in Fig. 7 where the meridians $m_1, \ldots, m_r$ are also shown.

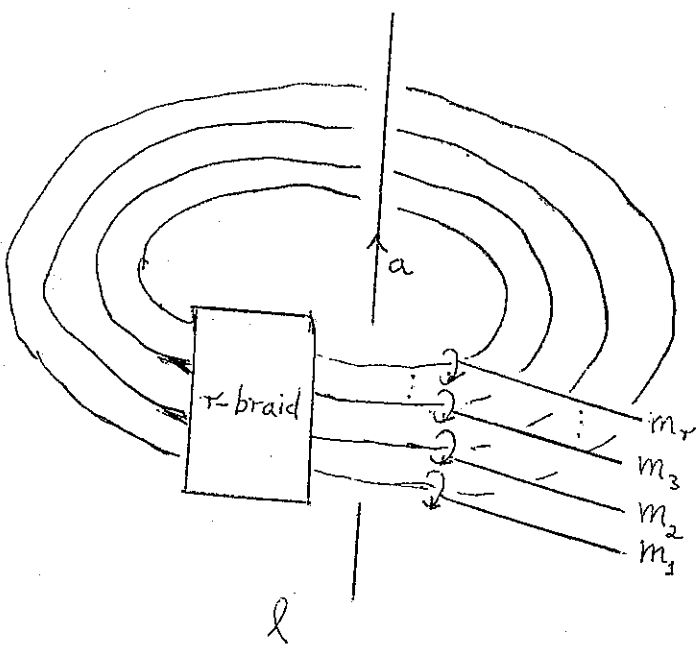


Fig. 7a

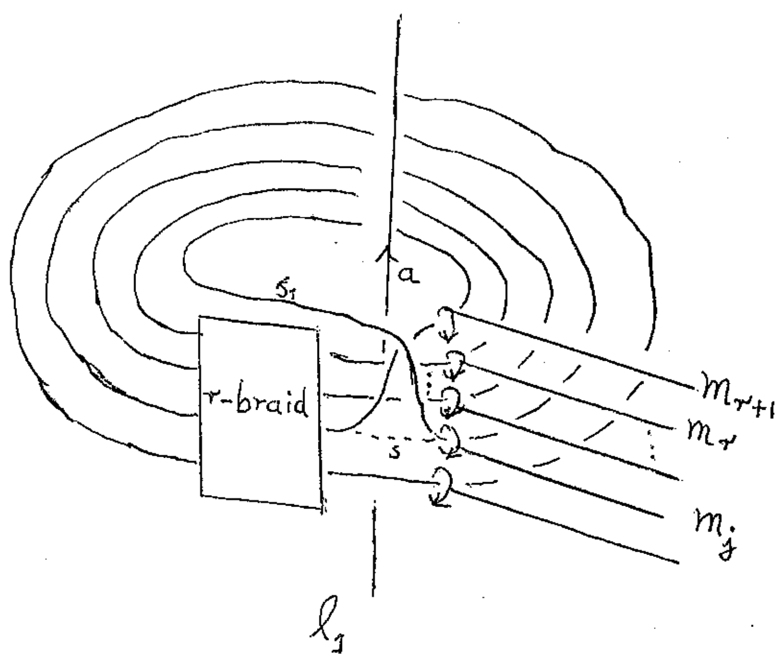


Fig. 7b

Since in $\pi_1(S^3 - l, \infty), [m_1 m_2 \ldots m_r] = [a]$ where a is the axis, which we think of as a loop based at ∞, $\omega([m_1])\omega([m_2]) \cdots \omega([m_r]) = \omega([a])$. Let d

be the number of cycles (including 1-cycles) of $\omega([a])$. If $d > 1$ then there is a j such that $\omega([m_j]) = (uv)$ where u and v belong to different cycles of $\omega([a])$ since $[m_1], \ldots, [m_r]$ generate $\pi_1(S^3 - l)$ and ω is transitive. Let $l_1 = (l - s) \cup s_1$ where s is a small arc in the $j - th$ string of l and s_1 is an arc which goes around the axis a as indicated in Fig. 7b (a Markov move). The arcs s, s_1 are such that $s \cup s_1$ is the boundary of a polyhedral 2-disk whose intersection with l is s and which does not intersect the (images of) the meridians $m_1, \ldots, m_r$. Then l_1 is a closed $(r+1)$-braid which is equivalent to l by a homeomorphism h which is the identity on $m_1, \ldots, m_r$. Hence, if we define ω_1 as the composition $\pi_1(S^3 - l_1, \infty) \xrightarrow{h_*} \pi_1(S^3 - l, \infty) \xrightarrow{\omega} S_q$, then $M(l_1, \omega_1) \approx M(l, \omega) \approx M^3, \omega_1([m_i]) = \omega([m_i]) \quad i = 1, \ldots, r$ and $\omega_1([m_{r+1}] = \omega([m_j])$ where m_{r+1} is the meridian shown in Fig. 7b. But now $\omega_1([a]) = \omega_1([m_1]) \cdots \omega_1([m_r])\omega_1([m_{r+1}]) = \omega([a]) \cdot (uv)$ has $d-1$ cycles.

If we repeat the process and make $d-1$ suitable Markov mores we finally obtain a closed braid l' and a representation $\omega'\colon \pi_1(S^3 - l', \infty) \to S_q$ such that $\omega'([a])$ has only one cycle and $M(l', \omega') \approx M^3$. Then, if $p\colon M(l', \omega') \to S^3$ is the branched covering map and $\{W_z\}$ is the standard book decomposition of S^3 having a as binding and disks W_z transversal to l' as pages we have that $\{p^{-1}(W_z)\}$ is a book decomposition of $M(l', \omega')$ with connected binding $p^{-1}(a)$. ∎

A corollary pointed out by Jonathan Simon is

Theorem 5 (Bing). *A closed connected 3-manifold M is S^3 if every simple closed curve in M is contained in a polyhedral 3-disk.*

Proof: The hypothesis implies that M is 1-connected. Let k be a fibered knot in M which, by hypothesis is contained in the interior of a disk D^3.

Since M is 1-connected $\overline{M-D^3}$ is a homotopy 3-disk contained in $M-k$. But $M-k$ is irreducible since it fibers over S^1. It follows that $\overline{M-D^3}$ is a 3-disk and $M \approx S^3$. ∎

Question. Can one give a proof "a la Alexander" of the fact that every orientable closed M^3 is a book of type $(0,d)$?

If so one would have another proof of Lickorish's theorem (every orientable closed M^3 can be obtained by Milnor-Wallace surgery on a link l in S^3; furthermore l can be taken to be a pure link).

Conjecture 1. Every nonorientable closed 3-manifold M^3 is an open book.

If the conjecture is true then, using the plumbing Lemma, M^3 has a fibered link with a prescribed number $d \geq 1$ of components and a codimension 1 foliation with precisely d compact leaves.

If W is a 2-disk with n holes and $h\colon W \to W$ is a homeomorphism which is the identity on ∂W then $\pi_1(M(h))$ has a presentation $(x_1, \dots, x_n \colon r_1, \dots, r_n)$ where $x_i = [s_i a_i s_i^{-1}] \in \pi_1(W) = F_n, r_i = [h(s_i) \cdot s_i^{-1}]$

s_i and a_i being the paths and loops indicated in Figure 8

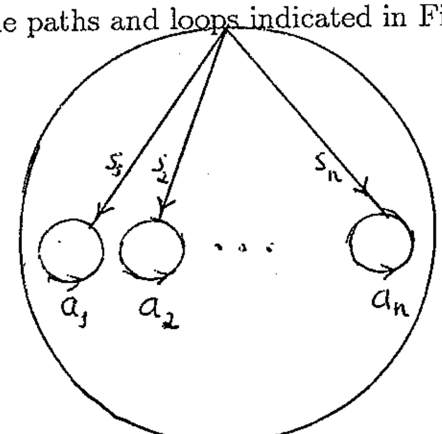


Fig. 8

One can see that the r_i satisfy

$$(A) \prod_{i=1}^{n} r_i x_i r_i^{-1} = \prod_{i=1}^{n} x_i \quad \text{in} \quad F$$

Conversely if $\mathcal{A} = (x_1, \ldots, x_n : r_1, \ldots, r_n)$ is such that (A) is satisfied (we call such an $\mathcal{A}$ an Artin n-presentation) then it follows from [Artin 1925] that there is a (unique up to isotopy rel ∂W) homeomorphism $h: W \to W$ of a disk with n holes onto itself such that $h \mid \partial W =$ identity and $[h(s_i) \cdot s_i^{-1}] = r_i \quad i = 1, \ldots, n$ so that, if we write $M(h) = M_{\mathcal{A}}, \pi_1(M_{\mathcal{A}}) = \mid \mathcal{A} \mid$ (the group presented by $\mathcal{A}$).

This yields the following algebraic characterization of 3-manifold groups.

Theorem 6. *G is the fundamental group of a closed orientable M^3 iff it has an Artin n-presentation for some n.*

Another albebraic characterization can be found in Birman [Bull. Australian...]. The uniqueness of Heegard splittings of S^3, proved by Waldhaussen, has allowed the formulation of the P.C. (Poincaré Conjecture) in purely algebraic terms. (See [Birman], [Traeub], [Jaco]). Using again Waldhaussen's Theorem and open books with planar pages one can give another algebraic equivalent of P.C.

Definition 1. Let $\mathcal{S}_n = \mid x_1, y_1, \ldots, x_n, y_n : \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i^{-1} x_i y_i \mid$ and let $N \subset \mathcal{S}_n$ be the normal closure of $y_1, \ldots, y_n$ in $\mathcal{S}_n$.

If $\mathcal{A}$ is an Artin n-presentation define the automorphism $\varphi_{\mathcal{A}} : \mathcal{S}_n \to \mathcal{S}_n$ by $\varphi_{\mathcal{A}}(x_i) = r_i x_i r_i^{-1}, \varphi_{\mathcal{A}}(y_i) = r_i y_i \quad i = 1, \ldots, n$.

Two Artin n-presentations $\mathcal{A}, \mathcal{A}'$ are equivalent ($\mathcal{A} \sim \mathcal{A}'$) if there exist automorphisms E_1, E_2 of $\mathcal{S}_n$ such that $E_1 \circ \varphi_{\mathcal{A}} = \varphi_{\mathcal{A}'} \circ E_2$ and $E_i(N) = N \quad i = 1, 2$.

One can see that $\mathcal{A} \sim \mathcal{A}' \Rightarrow M_{\mathcal{A}} \approx M_{\mathcal{A}'} \Rightarrow \mid \mathcal{A} \mid \approx \mid \mathcal{A}' \mid$. P.C. is then equivalent to

Conjecture 2. If A is an Artin n-presentation such that $\mid \mathcal{A} \mid = 1$ then $\mathcal{A} \sim \mathcal{T}$ where:

$$\mathcal{T} = (x_1, \dots, x_n : x_1, \dots, x_n).$$

This conjecture is true for $n \leq 2$.

One can also show that P.C. is true in the class of books of type $(1, 1)$.

We now give a formula for the μ invariant of a closed 3-manifold obtained by Milnor-Wallace surgery on a link. This can be applied to compute the μ invariant of an open book with planar pages since they are the manifolds obtained by Milnor-Wallace surgery on a pure link.

Suppose you have an oriented link $L = L_1 \cup \dots \cup L_n$ in S^3 with integer coefficients, i.e. an integer l_i is assigned to the component L_i of L.

Let M^3 be obtained by surgery on L, i.e. $M^3 = \chi(\varphi_1, \dots, \varphi_n)$ where $\varphi_i \colon S^1 \times D^2 \to S^3$ is an embedding sending $S^1 \times 0$ to L_i and $S^1 \times 1$ $(1 \in \partial D^2)$ to a curve having linking number l_i with L_i $\quad (i = 1, \dots, n)$.

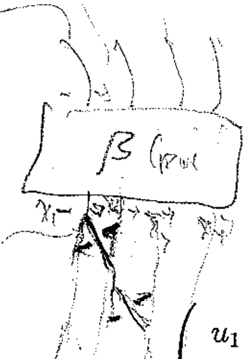
Let $Q = (l_{ij})$ be the $n \times n$ matrix such that

$$l_{ij} = \begin{cases} lk(L_i, L_j) \text{ if } i \neq j \\ l_i \text{ if } i = j \end{cases}$$

where lk denotes linking number.

(Q defines the quadratic form of a 1-connected 4-manifold whose boundary is M^3).

Assume M^3 is a Z_2-homology sphere, i.e. Q_2 the reduction of Q mod 2, is



nonsingular. Let U be the set of integers such that $u_i \neq 0$ where $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = Q_2^{-1} \cdot \begin{pmatrix} \bar{l}_1 \\ \vdots \\ \bar{l}_n \end{pmatrix}$ where $\bar{l}_i = l_i \bmod 2$. One can see that $L_U = \bigcup_{i \in U} L_i$ is a proper link (i.e. for every component L_i of L_U $lk(L_i, L_U - L_i)$ is even) so that the Arf invariant $\chi(L_U)$ of L_U is defined (one defines $\chi(\emptyset) = 0$). [References: "Dehn's construction . . ." section on slice links in the weak sense, or Kaufmann's "An invariant of link cobordism".] ($\chi(L_U)$ can be computed as follows: Find oriented bands $B_1, \ldots, B_{r-1}$ in S^3, r being the number of components of L_U, such that $L_U + \partial B_1 + \ldots + \partial B_{r-1}$ is a knot k, where $+$ denotes homology sum. Then $\chi(L_U) = \begin{cases} 0 & \text{if } \det k \equiv \pm 1 \mod 8 \\ \frac{1}{2} & \text{if } \det k \equiv \pm 3 \mod 8 \end{cases}$ $= \frac{d^2-1}{16}$

Finally $\mu(M^3) = \frac{\sigma(Q) - \sum_{i,j \in U} l_{ij}}{16} + \chi(L_U) \bmod 1$ where $\sigma(Q)$ is the signature of Q.

The proof is not difficult using Robertello's paper.

$\mu(M^3) = 1 \in \mathbb{Z}_2$

$\mu(M^3) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$

$Q = I$

$\mu(M^3) = \frac{1}{16}\left(\sigma(Q) - \sum_{i,j \in U} l_{ij} + d^2 - 1\right)$ $\quad d = \Delta(-1)$