

Extreme nonuniqueness of end-sum

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We give explicit examples of pairs of one-ended, open 4-manifolds whose end-sums yield uncountably many manifolds with distinct proper homotopy types. This answers strongly in the affirmative a conjecture of Siebenmann regarding nonuniqueness of end-sums. In addition to the construction of these examples, we provide a detailed discussion of the tools used to distinguish them; most importantly, the end-cohomology algebra. Key to our Main Theorem is an understanding of this algebra for an end-sum in terms of the algebras of summands together with ray-fundamental classes determined by the rays used to perform the end-sum. Differing ray-fundamental classes allow us to distinguish the various examples, but only through the subtle theory of infinitely generated abelian groups. An appendix is included which contains the necessary background from that area.

Keywords: End-sum; connected sum at infinity; end-cohomology; proper homotopy; direct limit; infinitely generated abelian group.

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1. Introduction

Our primary goal is a proof of the following theorem, which emphatically affirms a conjecture of Siebenmann [4, p. 1805] addressed in an earlier paper by the first and third authors of this paper [3].

Main Theorem. *There exist one-ended, open 4-manifolds M and N such that the end-sum of M and N yields uncountably many manifolds with distinct proper homotopy types.*

In addition to definitions, background, and proofs, we carefully develop the tools needed to distinguish between the aforementioned manifolds. Foremost among these is the end-cohomology algebra of an end-sum. We also discuss some intriguing open questions.

End-sum is a technique for combining a pair of noncompact n -manifolds in a manner that preserves the essential properties of the summands. Sometimes called *connected sum at infinity* in literature, end-sum is the natural analog of both the classical connected sum of a pair of n -manifolds and the boundary connected sum of a pair of n -manifolds with boundaries. The earliest intentional use of the end-sum operation appears to have been by Gompf [13] in his work on smooth structures on \mathbb{R}^4 . Other applications to the study of exotic \mathbb{R}^4 's can be found in Bennett [1] and Calcut and Gompf [2]. End-sum has also been useful in studying contractible n -manifolds not homeomorphic to \mathbb{R}^n . This is due to the fact that, unlike with classical connected sums, the end-sum of a pair of contractible manifolds is again contractible. For a sampling of such applications in dimension 3, see Myers [26] and Tinsley and Wright [32]; in dimension 4, see Calcut and Gompf [2] and Sparks [31]; and in dimensions $n \geq 4$, see Calcut, King, and Siebenmann [4]. For “incidental” applications of end-sum to the study of contractible open manifolds of dimension $n \geq 4$, see Curtis and Kwun [6] and Davis [7]. These incidental (unintentional) applications are due to the fact that the interior of a boundary connected sum may also be viewed as an end-sum of the corresponding interiors.

Each variety of connected sum involves arbitrary choices that lead to questions of well-definedness. For example, to perform a classical *connected sum* in the smooth category,^a one begins with a pair of smooth, connected, oriented n -manifolds, then chooses smooth n -balls $B_1 \subset \text{Int } M$ and $B_2 \subset \text{Int } N$ and an orientation reversing diffeomorphism $f : \partial B_1 \rightarrow \partial B_2$. From there, one declares $M \# N$ to be the oriented manifold $(M - \text{Int } B_1) \cup_f (N - \text{Int } B_2)$. Provided M and N are connected, standard but nontrivial tools from differential topology can be used to verify that, up to diffeomorphism, $M \# N$ does not depend upon the choices made. See [21, p. 90] for details. Note that well-definedness fails if one omits the connectedness hypothesis or ignores orientations.

^aSimilar definitions, conventions, and arguments allow for analogous connected sum operations in the piecewise linear and topological categories. For the sake of simplicity and focus, we will restrict our attention to the smooth category.

For smooth, oriented n -manifolds M and N with non-empty boundaries, a *boundary connected sum* is performed by choosing smooth $(n - 1)$ -balls $B_1 \subset \partial M$ and $B_2 \subset \partial N$, and an orientation reversing diffeomorphism $f : B_1 \rightarrow B_2$. Provided ∂M and ∂N are connected, an argument similar to the one used for ordinary connected sums shows that the adjunction space $M \cup_f N$ (suitably smoothed and oriented) is well defined up to diffeomorphism; it is sometimes denoted $M \diamond N$. Again, see [21, p. 97] for details.

An *end-sum* of a pair of smooth, oriented, noncompact n -manifolds M and N begins with the choice of properly embedded rays $r \subset \text{Int } M$ and $s \subset \text{Int } N$ and regular neighborhoods νr and νs of those rays. The regular neighborhoods are diffeomorphic to closed upper half-space \mathbb{R}_+^n , so each has boundary diffeomorphic to \mathbb{R}^{n-1} . Choose an orientation reversing diffeomorphism $f : \partial \nu r \rightarrow \partial \nu s$ to obtain the end-sum defined by $(M - \text{Int } \nu r) \cup_f (N - \text{Int } \nu s)$; sometimes this end-sum is denoted informally as $M \natural N$. By an argument resembling those used above, neither the choice of thickenings of r and s (that is, the regular neighborhoods νr and νs) nor the diffeomorphism f affect the diffeomorphism type of $M \natural N$. However, the choices of rays r and s are another matter. For example, if M has multiple ends, then rays in M tending to different ends of M can yield inequivalent end-sums, even in the simple $n = 2$ case. For that reason, we focus on the most elusive case where M and N are one-ended. The existence of knotted rays in 3-manifolds poses problems unique to that dimension. Indeed, Myers has exhibited an uncountable collection of topologically distinct end-sums where both summands are \mathbb{R}^3 . So, quickly we arrive at the appropriate question: *For smooth, oriented, one-ended, open n -manifolds M and N where $n \geq 4$, is end-sum well defined up to diffeomorphism?* In many cases the answer is affirmative. For example, $\mathbb{R}^4 \natural \mathbb{R}^4$ is always \mathbb{R}^4 [14]. More generally, the end-sum of n -manifolds with Mittag-Leffler ends and $n \geq 4$ depends only on the chosen ends [2]. Nevertheless, Siebenmann conjectured that counterexamples should exist in general [4, p. 1805]. His suspicion was confirmed by Calcut and Haggerty [3] where, for numerous pairs of smooth, one-ended, open 4-manifolds, it was shown that end-sums can produce non-diffeomorphic (in fact, non-proper homotopy equivalent) manifolds. Here, we will refine the techniques employed there to significantly extend that work.

As in the earlier work, the primary tool used to distinguish between various noncompact n -manifolds will be their end-cohomology algebras — more specifically it is the ring structure of that algebra that holds the key. This is an essential point since every end-sum herein of a given pair of one-ended manifolds has homology and cohomology groups (absolute and “end”) in each dimension that are isomorphic to those of any other end-sum of the same two manifolds. To allow for differences in these end-cohomology algebras, it will be necessary to work with manifolds that have substantial cohomology at infinity. That leads us naturally to the well studied, but subtle, area of infinitely generated abelian groups. For the benefit of the reader with limited background in that area, we have included an appendix with key definitions and proofs of the fundamental facts used in this paper. Capturing this

subtle algebra in the form of a manifold requires some care — most significantly, a precise description of the end-cohomology algebra of an end-sum in terms of the end-cohomology algebras of the summands with input from so-called ray-fundamental classes determined by the chosen rays. We provide a careful development of this topic, as suggested to us by Henry King.

Given past applications of end-sum, the following open question deserves attention.

Question 1.1. For contractible, open n -manifolds M and N of dimension $n \geq 4$, is $M \natural N$ well-defined up to diffeomorphism or up to homeomorphism?

Note that, by Poincaré duality “at the end” (see [12, p. 361]), the end-cohomology algebra of a contractible, open n -manifold is isomorphic to the ordinary cohomology algebra of S^{n-1} . So, the methods used in this paper appear to be of no use in attacking this problem.

Using ladders based on exotic spheres, Calcut and Gompf [2, Example 3.4(a)] gave pairs of smooth, one-ended, open n -manifolds for some $n \geq 7$ whose end-sums are piecewise linearly homeomorphic but not diffeomorphic. It is unknown whether examples exist in dimension $n = 4$ whose end-sums are homeomorphic but not diffeomorphic. A key open question is the following (see [3, p. 3282; 2, p. 1303]).

Question 1.2. Can the (oriented) end-sums of a smooth, oriented, one-ended, open 4-manifold M with a fixed oriented exotic \mathbb{R}^4 be distinct up to diffeomorphism?

If such examples exist, then it appears that distinguishing them will be difficult [2, Proposition 5.3].

The outline of this paper is as follows. Section 2 lays out some conventions, defines end-sum, and discusses end-cohomology. Section 3 defines some manifolds (stringers, surgered stringers, and ladders) useful for our purposes and computes their end-cohomology algebras. Section 4 classifies stringers, surgered stringers, and ladder manifolds based on closed surfaces. Section 5 defines ray-fundamental classes and presents a proof of an unpublished result of Henry King that computes the end-cohomology algebra of a binary end-sum. Section 6 computes ray-fundamental classes in surgered stringers and ladders. Section 7 proves the Main Theorem. Appendix A presents some relevant results from the theory of infinitely generated abelian groups.

2. Conventions, End-Sum, and End-Cohomology

2.1. Conventions

Throughout this paper, topological spaces are metrizable, separable, and locally compact. In particular, each space has a compact exhaustion (see Sec. 2.3). Our focus is on one-ended examples, but many of the foundational results established

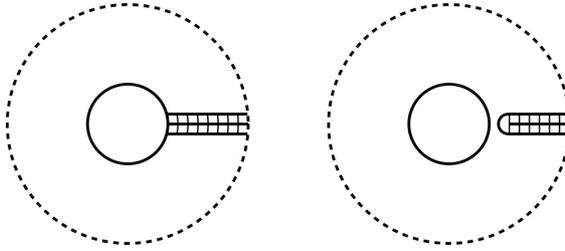


Fig. 1. Neatly embedded ray $r \subset M$ and a smooth closed tubular neighborhood $\tau r \subset M$ (left), and ray $r \subset \text{Int } M$ and a smooth closed regular neighborhood $\nu r \subset \text{Int } M$ (right).

here apply to manifolds with arbitrarily many ends. Unless explicitly stated otherwise, manifolds are smooth, connected, and oriented. We follow the orientation conventions of Guillemin and Pollack [16, Chap. 3]. In particular, the boundary ∂M of a manifold M is oriented by the outward normal first convention. Let $\text{Int } M$ denote the manifold interior of M . A manifold without boundary is *closed* if it is compact and is *open* if it is noncompact. A map of spaces is *proper* provided the inverse image of each compact set is compact. A *ray* is a smooth proper embedding of the real half-line $[0, \infty)$. The submanifold $[0, \infty) \subset \mathbb{R}$ is standardly oriented [16, Chap. 3]. By $M \approx N$ we indicate diffeomorphic manifolds (not necessarily preserving orientation).

We will consider rays in manifold interiors as well as neatly embedded rays. Recall that a manifold A embedded in a manifold B is said to be *neatly embedded* provided A is a closed subspace of B , $\partial A = A \cap \partial B$, and A meets ∂B transversely (see [18, p. 30; 21, pp. 27–31, 62]). The closed subspace condition is automatically satisfied by any proper embedding. Now, let $r \subset M$ be a neatly embedded ray. We let $\tau r \subset M$ denote a smooth closed tubular neighborhood of r in M as in Fig. 1 (left). By definition, a *closed tubular neighborhood* is a restriction of an open tubular neighborhood (see [18, pp. 109–118; 21, pp. 46–53]); we will always assume that closed tubular neighborhoods are restrictions of neat tubular neighborhoods. In particular, the disk bundle τr over r meets ∂M in exactly the disk over the endpoint 0. Closed tubular neighborhoods of r in M are unique up to ambient isotopy fixing r . Next, let $r \subset \text{Int } M$ be a ray. We let $\nu r \subset \text{Int } M$ denote a smooth closed regular neighborhood of r in $\text{Int } M$ as in Fig. 1 (right). Existence and ambient uniqueness of smooth closed tubular neighborhoods and collars imply the same results for smooth closed regular neighborhoods [4, pp. 1815].

2.2. End-sum

We now define the end-sum of two noncompact manifolds. An *end-sum pair* (M, r) consists of a smooth, oriented, connected, noncompact manifold M together with a ray $r \subset \text{Int } M$. We allow M to have arbitrarily many ends. Consider two end-sum pairs (M, r) and (N, s) where M and N have the same dimension $m \geq 2$. The *end-sum* of (M, r) and (N, s) , which we denote by $(M, r) \natural (N, s)$, is defined as follows. Choose smooth closed regular neighborhoods $\nu r \subset \text{Int } M$ and $\nu s \subset \text{Int } N$ of r and

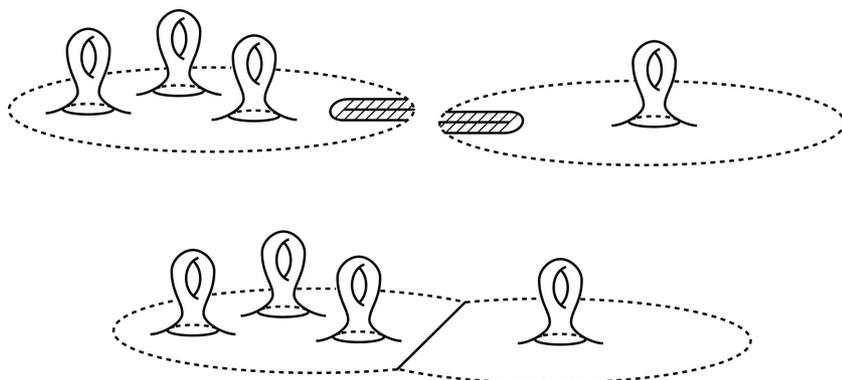


Fig. 2. End-sum of two manifold/ray pairs.

s , respectively. Delete the interiors of these regular neighborhoods and glue the resulting manifolds $M - \text{Int } \nu r$ and $N - \text{Int } \nu s$ along their boundaries $\partial \nu r \approx \mathbb{R}^{m-1}$ and $\partial \nu s \approx \mathbb{R}^{m-1}$ by an orientation reversing diffeomorphism as in Fig. 2.

Remark 2.1. The manifold $(M, r) \natural (N, s)$ is smooth and oriented, and its diffeomorphism type is independent of the choices of the regular neighborhoods and the glueing diffeomorphism [3, Sec. 2]. While this binary end-sum is sufficient for our purposes, we mention that it is a special case of a more general operation that also applies to piecewise linear and topological manifolds and allows countably many summands [4]. Alternatively, one may view the end-sum operation as the addition of a so-called 1-handle at infinity [2].

2.3. End-cohomology

Throughout, R denotes a commutative, unital ring. We use the singular theory for ordinary (co)homology. We suppress the coefficient ring when that ring is \mathbb{Z} .

We will distinguish noncompact manifolds by the isomorphism types of their (graded) end-cohomology algebras. Just as cohomology is a homotopy invariant of spaces, end-cohomology is a proper homotopy invariant of spaces. We adopt the direct limit approach to end-cohomology. An alternative may be found in several places including Conner [5], Raymond [27], Massey [23, Chap. 10], and Geoghegan [12, Chap. 12]. The alternative approach provides some advantages in terms of establishing the foundations of end-cohomology and comparing it to other cohomology theories. On the other hand, we find the direct limit approach invaluable for carrying out concrete calculations. For the benefit of the reader — and since the arguments are straightforward and satisfying — we take the time to develop the basics of end-cohomology straight from the direct limit definition.^b

^bFor background on proper homotopy, see [15, pp. 58–59; 20, Chap. 3]. For background on direct systems and direct limits, see [9, Chap. 8; 23, Appendix; 28, Chap. 6.9].

Fix a topological space X . Define the poset (\mathcal{K}, \leq) where \mathcal{K} is the set of compact subsets of X and $K \leq K'$ means $K \subseteq K'$. We have a direct system of graded R -algebras $H^*(X - K; R)$ where $K \in \mathcal{K}$. The morphisms of this direct system are restrictions induced by inclusions. Define $H_e^*(X; R)$, the *end-cohomology algebra*, to be the direct limit of this direct system. For the relative version, let (X, A) be a *closed pair*, namely a space X together with a closed subspace $A \subseteq X$. Regard X as the closed pair (X, \emptyset) . Consider the direct system $H^*(X - K, A - K; R)$ where $K \in \mathcal{K}$ and the morphisms are restrictions. Define $H_e^*(X, A; R)$ to be the direct limit of this direct system. Similarly, reduced end-cohomology $\tilde{H}_e^*(X, A; R)$ is the direct limit of the direct system $\tilde{H}^*(X - K, A - K; R)$.

We employ a standard explicit model of the direct limit [9, p. 222] where an element of $H_e^*(X, A; R)$ is represented by an element of $H^*(X - K, A - K; R)$ for some compact K . Two representatives $\alpha \in H^*(X - K, A - K; R)$ and $\alpha' \in H^*(X - K', A - K'; R)$ are equivalent if they have the same restriction in some $H^*(X - K'', A - K''; R)$, where $K, K' \subseteq K''$.

Recall that a *compact exhaustion* of X is a nested sequence $K_1 \subseteq K_2 \subseteq \dots$ of compact subsets of X whose union equals X and where $K_j \subseteq K_{j+1}^\circ$ for each j . Here, K_{j+1}° denotes the topological interior of K_{j+1} as a subspace of X . By our hypotheses on spaces, each space has a compact exhaustion (see [19, p. 75]). Let $\{K_j\}$ be any compact exhaustion of X . As $\{K_j\}$ is cofinal in \mathcal{K} , we may compute $H_e^*(X, A; R)$ using the direct system indexed by $\mathbb{Z}_{>0}$. Namely, there is a canonical isomorphism (see [9, p. 224])

$$H_e^*(X, A; R) \cong \varinjlim H^*(X - K_j, A - K_j; R). \tag{2.1}$$

We claim that we may delete instead the topological interior K_j° of K_j to obtain the canonical isomorphism

$$H_e^*(X, A; R) \cong \varinjlim H^*(X - K_j^\circ, A - K_j^\circ; R). \tag{2.2}$$

To prove the claim, we show that the right-hand sides of (2.1) and (2.2) are canonically isomorphic. Let G_j and G'_j denote the j th terms in these direct systems. The inclusions $K_1^\circ \subseteq K_1 \subseteq K_2^\circ \subseteq K_2 \subseteq \dots$ induce the obvious maps between these direct systems and give the commutative diagram

$$\begin{array}{ccccccc}
 G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & \dots \\
 \uparrow & \searrow & \uparrow & \searrow & \uparrow & \searrow & \\
 G'_1 & \longrightarrow & G'_2 & \longrightarrow & G'_3 & \longrightarrow & \dots
 \end{array} \tag{2.3}$$

We get induced maps $\phi : \varinjlim G_j \rightarrow \varinjlim G'_j$ and $\psi : \varinjlim G'_j \rightarrow \varinjlim G_j$ between direct limits [9, p. 223]. It is a simple exercise to prove that $\psi \circ \phi$ and $\phi \circ \psi$ are the respective identity maps (use Eilenberg and Steenrod [9, pp. 220–223]). This proves the claim. Passing to a subsequence in either (2.1) or (2.2) canonically preserves the isomorphism type of the direct limit since these isomorphisms are independent of the choice of compact exhaustion (see also [9, p. 224]).

Proof. By hypothesis, $g \circ f$ is proper homotopy equivalent to id_X by a proper homotopy sending A into B at all times, and similarly for $f \circ g$ and id_Y . By Lemma 2.2 and the preceding observations, $f_e^* \circ g_e^* = \text{id}$ and $g_e^* \circ f_e^* = \text{id}$. \square

Lemma 2.4. *For each closed pair (X, A) there is the induced long exact sequence*

$$\cdots \rightarrow H_e^k(X, A; R) \rightarrow H_e^k(X; R) \rightarrow H_e^k(A; R) \rightarrow H_e^{k+1}(X, A; R) \rightarrow \cdots.$$

Proof. Let $\{K_j\}$ be a compact exhaustion of X . As A is closed in X , $\{A \cap K_j\}$ is a compact exhaustion of A . Consider the biinfinite commutative diagram whose j th column is the long exact sequence for the pair $(X - K_j, A - K_j)$. The rows in this diagram are the various direct systems $H^k(A - K_j; R)$, $H^k(X - K_j; R)$, and $H^k(X - K_j, A - K_j; R)$. The maps in this diagram between successive rows induce maps of their direct limits. The resulting sequence of direct limits is exact since the direct limit is an exact functor in the category of R -modules (see [9, p. 225] or [23, p. 389]). \square

A *closed triple* (X, A, B) is a space X together with subspaces $B \subseteq A \subseteq X$ each closed in X . With the long exact sequences for the closed pairs (A, B) , (X, B) , and (X, A) in hand, a well-known diagram chase [9, p. 24] proves the following.

Corollary 2.5. *For each closed triple (X, A, B) there is the induced long exact sequence*

$$\cdots \rightarrow H_e^k(X, A; R) \rightarrow H_e^k(X, B; R) \rightarrow H_e^k(A, B; R) \rightarrow H_e^{k+1}(X, A; R) \rightarrow \cdots.$$

Remark 2.6. It is crucial for end-cohomology that one consider *closed* pairs and triples. Otherwise, one does not obtain induced maps for the usual long exact sequences, and the direct system $H^*(A \cap K_j; R)$ (where $\{K_j\}$ is a compact exhaustion of X) need not compute $H_e^k(A; R)$.

An *excisive triad* $(X; A, B)$ is a space X together with two closed subspaces $A \subseteq X$ and $B \subseteq X$ such that $X = A^\circ \cup B^\circ$ where A° and B° are the topological interiors of A and B in X , respectively.^c

Lemma 2.7. *Let $(X; A, B)$ be an excisive triad and set $C = A \cap B$. Then the inclusion $\phi : (A, C) \rightarrow (X, B)$ induces the excision isomorphism*

$$\phi^* : H_e^*(X, B; R) \rightarrow H_e^*(A, C; R).$$

Proof. Let $\{K_j\}$ be a compact exhaustion of X . So, $\{A \cap K_j\}$, $\{B \cap K_j\}$, and $\{C \cap K_j\}$ are compact exhaustions of A , B , and C , respectively. We have the two

^cNote the subtle notational distinction between a triple and a triad.

direct systems

$$\begin{aligned} H^*(A - K_j, C - K_j; R), \\ H^*(X - K_j, B - K_j; R), \end{aligned} \tag{2.5}$$

where the morphisms in both systems are induced by inclusions. For each j , we have the inclusion

$$\phi_j : (A - K_j, C - K_j) \rightarrow (X - K_j, B - K_j).$$

Observe that $X - K_j = (A - K_j)^\circ \cup (B - K_j)^\circ$ where $(A - K_j)^\circ$ denotes the topological interior of $A - K_j$ as a subspace of $X - K_j$ and similarly for $(B - K_j)^\circ$. Therefore, each ϕ_j^* is an excision isomorphism on ordinary R -cohomology. By [9, p. 223], these isomorphisms induce an isomorphism between the direct limits of the direct systems (2.5). Two applications of (2.1) now complete the proof. \square

The following corollary is useful (compare Eilenberg and Steenrod [9, p. 32] and May [24, pp. 145]).

Corollary 2.8. *Let $(X; A, B)$ be an excisive triad and set $C = A \cap B$. Denote the inclusion maps by $i_A : (A, C) \hookrightarrow (X, C)$ and $i_B : (B, C) \hookrightarrow (X, C)$. Then the map*

$$h : H_e^*(X, C; R) \rightarrow H_e^*(A, C; R) \oplus H_e^*(B, C; R)$$

defined by $h(\alpha) = (i_A^(\alpha), i_B^*(\alpha))$ is a graded R -algebra isomorphism.*

Recall that the product is coordinatewise in the direct sum of algebras.

Proof. The commutative diagram of inclusions

$$\begin{array}{ccccc} (A, C) & & & & (B, C) \\ & \searrow i_A & & \swarrow i_B & \\ & & (X, C) & & \\ & \swarrow & & \searrow & \\ (X, B) & & & & (X, A) \end{array} \tag{2.6}$$

induces the commutative diagram

$$\begin{array}{ccccc} H_e^*(A, C; R) & & & & H_e^*(B, C; R) \\ & \swarrow i_A^* & & \nwarrow i_B^* & \\ & & H_e^*(X, C; R) & & \\ & \swarrow & & \nwarrow & \\ H_e^*(X, B; R) & & & & H_e^*(X, A; R) \end{array} \tag{2.7}$$

\cong exc. \uparrow \quad \uparrow \cong exc.

The vertical maps are excision isomorphisms (Lemma 2.7). Hence, the two lower maps are injective and the two upper maps are surjective. The two diagonals are

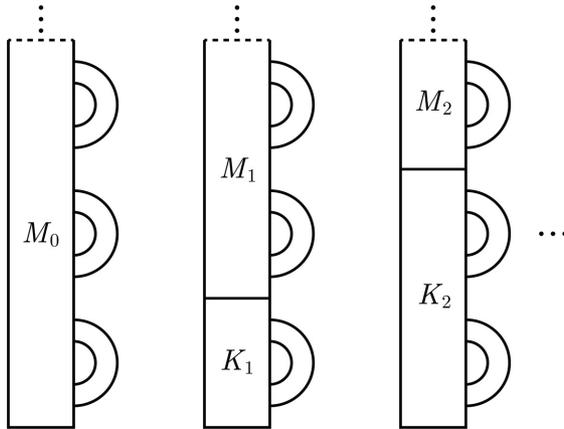


Fig. 3. Manifold M with a compact exhaustion $\{K_j\}$ and a (closed) neighborhood system of infinity $\{M_j\}$.

exact being portions of long exact sequences for triples (Corollary 2.5). These properties of (2.7) readily imply that h is both injective and surjective. \square

Remark 2.9. Let $r \subset \text{Int } M$ be a ray and $\nu r \subset \text{Int } M$ be a smooth closed regular neighborhood of r . Define $\widehat{M} := M - \text{Int } \nu r$. We claim that the inclusion $\phi : (\widehat{M}, \partial \nu r) \hookrightarrow (M, \nu r)$ induces an isomorphism ϕ_e^* on end-cohomology. However, the corresponding triad $(M; \widehat{M}, \nu r)$ is not excisive since M is not the union of the topological interiors \widehat{M}° and νr° of \widehat{M} and νr in M , respectively. This nuisance is easily fixed using a closed collar. Let $Z \approx \partial \nu r \times [0, 1]$ be a closed collar on $\partial \nu r$ in νr . Notice that ϕ equals the composition of the inclusions

$$(\widehat{M}, \partial \nu r) \xrightarrow{i} (\widehat{M} \cup Z, Z) \xrightarrow{j} (M, \nu r).$$

Both induced morphisms i_e^* and j_e^* are isomorphisms. The former holds since i is properly homotopic to the identity map on $(\widehat{M}, \partial \nu r)$ using the obvious proper strong deformation retraction that collapses the closed collar Z to $\partial \nu r$. The latter holds since j_e^* is the excision isomorphism from the excisive triad $(M; \widehat{M} \cup Z, \nu r)$. Hence, ϕ_e^* is an isomorphism and the claim is proved. Excision is used in Sec. 5 and Corollary 2.8 is used in the proof of Theorem 5.4. In each of these places, we leave the standard collaring fix to the reader.

For a general noncompact space or manifold, it appears to be difficult to compute the end-cohomology algebra in a comprehensible manner. So, we deliberately construct manifolds (stringers, surgered stringers, and ladders) with tractable algebras that fit into the following framework.

Let M be a connected space with a compact exhaustion $\{K_j\}$ where $j \in \mathbb{Z}_{\geq 0}$. Assume $K_0 = \emptyset$. Define $M_j := M - K_j^\circ$ where K_j° is the topological interior of K_j

as a subspace of M . So, each M_j is closed in M and

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$$

is a (closed) neighborhood system of infinity as in Fig. 3. By (2.2), we have $H_e^*(M; R) \cong \varinjlim H^*(M_j; R)$.

For each j , let $i_j : M_{j+1} \hookrightarrow M_j$ be the inclusion. Suppose that for each $j \in \mathbb{Z}_{\geq 0}$ there is a retraction $r_j : M_j \rightarrow M_{j+1}$ (in Fig. 3, the retraction r_j folds up the bottom of M_j). The composition $r_j \circ i_j$ equals the identity on M_{j+1} . So, $i_j^* \circ r_j^*$ equals the identity on $H^*(M_{j+1}; R)$ and each i_j^* is surjective. By [9, p. 222], each of the canonical morphisms

$$q_i : H^*(M_i; R) \rightarrow H_e^*(M; R)$$

is surjective with kernel Q_i equal to the submodule of elements that are eventually sent to 0 in the direct system $H^*(M_j; R)$. Here, $q_i(\alpha) := \llbracket \alpha \rrbracket$. Hence, for each $i \in \mathbb{Z}_{\geq 0}$ we have $H^*(M_i, R)/Q_i \cong H_e^*(M; R)$. This discussion applies to relative and reduced end-cohomology as well.

3. Stringers, Surgered Stringers, and Ladders

In this section, we define some manifolds and present their end-cohomology algebras. These will be used in our proof of the Main Theorem.

Let X be a closed, connected, oriented n -manifold with $n \geq 2$. The *stringer* based on X is $[0, \infty) \times X$ with the product orientation [16, Chap. 3]. Let $X_t = \{t\} \times X$, so the oriented boundary of the stringer is $-X_0$. The end-cohomology algebra of the stringer is

$$\widetilde{H}_e^*([0, \infty) \times X; R) \cong \widetilde{H}^*(X; R).$$

The *surgered stringer* $\mathbb{S}(X)$ based on X is obtained from the stringer on X by performing countably many oriented 0-surgeries as in Fig. 4.

We refer to the glued-in copies of $D^1 \times S^n$ as *rungs*. The space $X_0 \vee J$ in Fig. 4 is the wedge of X_0 and J , where J is the wedge of a ray, n -spheres S_j , and 1-spheres T_j . It is a strong deformation retract of $\mathbb{S}(X)$ by an argument similar to the one provided in [3, Lemma 3.2].

The surgered stringer $\mathbb{S}(X)$ is oriented using the orientation of the stringer $[0, \infty) \times X$. Let $\mathbb{S}_{[j,k]}$ denote the points of $\mathbb{S}(X)$ with heights in the interval $[j, k]$ as in Fig. 5. We orient $\mathbb{S}_{[j,k]}$ as a codimension-zero submanifold of $\mathbb{S}(X)$.

We orient each n -sphere S_j so that the oriented boundary of the cobordism $\mathbb{S}_{[j,j+1/2]}$ is $X_{j+1/2} - X_j + S_j$. Thus, the oriented boundary of $\mathbb{S}_{[j+1/2,j+1]}$ is $X_{j+1} - X_{j+1/2} - S_j$.

Let s^j denote the fundamental class $[S_j]$ of S_j , and let t^j denote the fundamental class $[T_j]$ of T_j . So, the nonzero reduced integer homology groups of J are $\widetilde{H}_n(J) \cong \mathbb{Z}[s]$ and $\widetilde{H}_1(J) \cong \mathbb{Z}[t]$. Define σ^j and τ^j to be the dual fundamental classes $[S_j]^*$

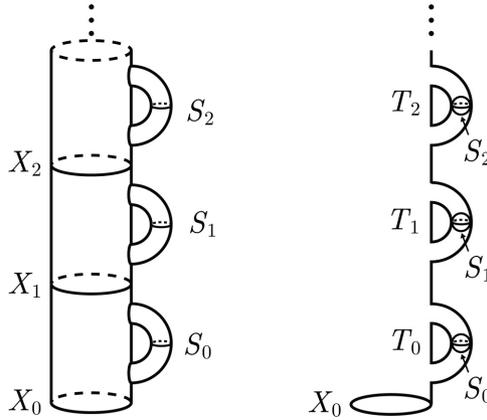


Fig. 4. Surgered stringer $\mathbb{S}(X)$ and a strong deformation retract $X_0 \vee J$ of $\mathbb{S}(X)$.

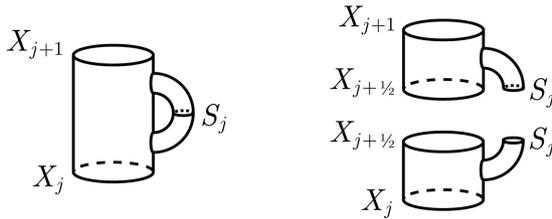


Fig. 5. Cobordisms $\mathbb{S}_{[j,j+1]}$, $\mathbb{S}_{[j,j+1/2]}$, and $\mathbb{S}_{[j+1/2,j+1]}$ in $\mathbb{S}(X)$.

and $[T_j]^*$ so that the nonzero reduced cohomology groups of J are

$$\begin{aligned} \tilde{H}^n(J; R) &\cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[s], R) \cong R[[\sigma]], \\ \tilde{H}^1(J; R) &\cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[t], R) \cong R[[\tau]]. \end{aligned}$$

All cup products in $\tilde{H}^*(J; R)$ vanish.

An argument similar, but simpler, to the one provided in [3, Sec. 3] now shows that the end-cohomology algebra of $\mathbb{S}(X)$ is

$$\tilde{H}_e^k(\mathbb{S}(X); R) \cong \begin{cases} H^n(X; R) \oplus R[[\sigma]]/R[\sigma] & \text{if } k = n, \\ H^k(X; R) \oplus 0 & \text{if } 2 \leq k \leq n - 1, \\ H^1(X; R) \oplus R[[\tau]]/R[\tau] & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The cup product is coordinatewise in the direct sum; it is that of X in the first coordinate and vanishes in the second coordinate.

Let X and Y be closed, connected, oriented n -manifolds with $n \geq 2$. The *ladder manifold* $\mathbb{L}(X, Y)$ based on X and Y is obtained from the stringers based on X and on Y by performing countably many oriented 0-surgeries as in Fig. 6 (Ladder manifolds were the primary objects of study in [3]. See that paper for more details).

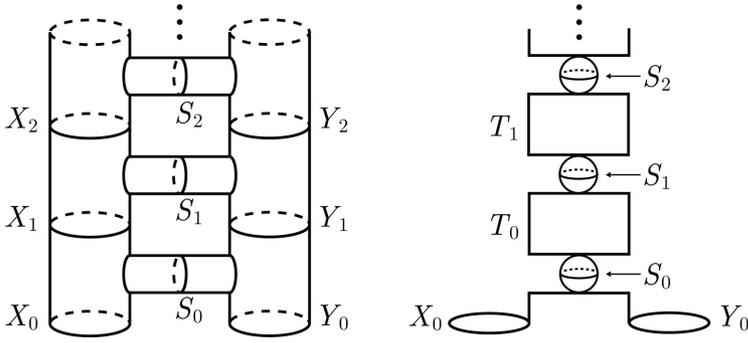


Fig. 6. Ladder manifold $\mathbb{L}(X, Y)$ and a strong deformation retract $X_0 \vee J \vee Y_0$.

Again, the glued-in copies of $D^1 \times S^n$ are called *rungs*. The ladder manifold is oriented using the orientations of the stringers based on X and Y . The oriented boundary of $\mathbb{L}(X, Y)$ is $-X_0 - Y_0$. Let $\mathbb{L}_{[j,k]}$ denote the points of $\mathbb{L}(X, Y)$ with heights in the interval $[j, k]$. We orient $\mathbb{L}_{[j,k]}$ as a codimension-zero submanifold of $\mathbb{L}(X, Y)$. The cobordism $\mathbb{L}_{[j,j+1]}$ is the union of two connected cobordisms with shared boundary component S_j . We orient each S_j so that the oriented boundaries of these cobordisms are $X_{j+1} - X_j + S_j$ and $Y_{j+1} - Y_j - S_j$. The ladder manifold $\mathbb{L}(X, Y)$ also contains 1-spheres T_j as shown in Fig. 6, and it strong deformation retracts to the wedge $X_0 \vee J \vee Y_0$ as explained in [3, p. 3287].

Let s^j denote the fundamental class $[S_j]$ of S_j , and let t^j denote the fundamental class $[T_j]$ of T_j . Again, the nonzero reduced integer homology groups of J are $\tilde{H}_n(J) \cong \mathbb{Z}[s]$ and $\tilde{H}_1(J) \cong \mathbb{Z}[t]$. Define σ^j and τ^j to be the dual fundamental classes $[S_j]^*$ and $[T_j]^*$ so that nonzero reduced cohomology groups of J are

$$\begin{aligned} \tilde{H}^n(J; R) &\cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[s], R) \cong R[[\sigma]], \\ \tilde{H}^1(J; R) &\cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[t], R) \cong R[[\tau]]. \end{aligned}$$

All cup products in $\tilde{H}^*(J; R)$ vanish. By [3, Sec. 3], the end-cohomology algebra of the ladder manifold $\mathbb{L}(X, Y)$ is

$$\tilde{H}_e^k(\mathbb{L}(X, Y); R) \cong \begin{cases} (H^n(X; R) \oplus R[[\sigma]] \oplus H^n(Y; R))/K & \text{if } k = n, \\ H^k(X; R) \oplus 0 \oplus H^k(Y; R) & \text{if } 2 \leq k \leq n - 1, \\ H^1(X; R) \oplus R[[\tau]]/R[\tau] \oplus H^1(Y; R) & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $K := \{(\sum \beta_i, \beta, -\sum \beta_i) \mid \beta = \sum \beta_i \sigma^i \in R[\sigma]\} \cong R[\sigma]$. The cup product is coordinatewise in the direct sum; it is that of X in the first coordinate, that of Y in the third coordinate, and vanishes in the second coordinate.

Remark 3.1. As X and Y are closed, connected, and oriented n -manifolds, we have that

$$\tilde{H}_e^n(\mathbb{L}(X, Y); R) \cong (R \oplus R[[\sigma]] \oplus R)/K.$$

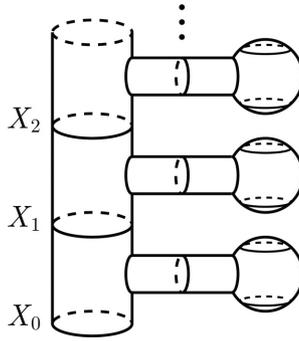


Fig. 7. Classical connected sum N of the stringer based on X with countably many $(n+1)$ -spheres.

When $R = \mathbb{Z}$, we show in Appendix A that the dual module of this \mathbb{Z} -module is isomorphic to \mathbb{Z} . On the other hand, for any ring R the canonical R -module homomorphism

$$R \oplus R \rightarrow (R \oplus R[[\sigma]] \oplus R)/K$$

defined by $(r, s) \mapsto [[(r, 0, s)]]$ is injective and, hence, an R -module isomorphism onto its image $(R \oplus 0 \oplus R)/K$. When R is a field, $(R \oplus 0 \oplus R)/K$ is a two-dimensional R -vector space. When $R = \mathbb{Z}$, $(R \oplus 0 \oplus R)/K$ is a rank two free \mathbb{Z} -module. For any ring R , each cup product with value of degree n must lie in $(R \oplus 0 \oplus R)/K$.

For many base manifolds, surgered stringers and ladder manifolds have nonisomorphic end-cohomology algebras. The proof of Theorem 4.2 shows various techniques for distinguishing these algebras. However, in some exceptional cases these manifolds have diffeomorphic ends.

Proposition 3.2. *Let X be a closed, connected, oriented n -manifold where $n \geq 1$. Let $M = \mathbb{L}(X, S^n) \cup_{\partial} D^{n+1}$ be the ladder manifold with the S^n boundary component capped by an $(n+1)$ -disk. Then M is diffeomorphic to $\mathbb{S}(X)$. In particular, $\mathbb{L}(X, S^n)$ and $\mathbb{S}(X)$ have diffeomorphic ends and, hence, isomorphic end-cohomology algebras.*

Proof. Let N be the (classical) connected sum of the stringer $[0, \infty) \times X$ and countably many $(n + 1)$ -spheres as in Fig. 7.

Note that $N \approx [0, \infty) \times X$. Performing countably many oriented 0-surgeries on N yields M , and performing them on $[0, \infty) \times X$ yields $\mathbb{S}(X)$. □

Remark 3.3. Given a manifold Y (not necessarily connected) and two proper disjoint rays in Y , *ladder surgery* is the operation where one performs countably many oriented 0-surgeries on Y using the 0-spheres given by the corresponding integer points on the rays. Properly homotopic rays yield diffeomorphic manifolds (see [2, Definition 3.1 and Corollary 4.13]).

Corollary 3.4. *For any ring R , there is an R -module isomorphism*

$$f : R \oplus R[[x]]/R[x] \rightarrow (R \oplus R[[x]] \oplus R)/K,$$

$$\left(r, \left[\sum_{i=0}^{\infty} c_i x^i \right] \right) \mapsto \left[\left(r - c_0, \sum_{i=1}^{\infty} (c_i - c_{i-1}) x^i, c_0 \right) \right].$$

Proof. The topological proof of Proposition 3.2 determines the function f . With f in hand, it is straightforward to verify (purely algebraically) that f is a well-defined R -module isomorphism. The inverse function f^{-1} is given by

$$\left[\left(r, \sum_{i=0}^{\infty} a_i x^i, s \right) \right] \mapsto \left(r + s, \left[\sum_{i=0}^{\infty} \left(s + \sum_{j=0}^i a_j \right) x^i \right] \right).$$

In particular, f maps $(1, [0]) \mapsto [(1, 0, 0)]$ and $(1, [\frac{1}{1-x}]) \mapsto [(0, 0, 1)]$. □

Consider the \mathbb{Z} -module $G = \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$. The submodule $0 \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$ is determined *algebraically* in an isomorphism invariant manner as the elements of G sent to 0 by every element of the dual \mathbb{Z} -module of G (see Corollary A.2). However, the submodule $\mathbb{Z} \oplus 0$ cannot be determined algebraically as shown by the next corollary.

Corollary 3.5. *Consider the \mathbb{Z} -module $G = \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$. The elements $(1, [0])$ and $(1, [\frac{1}{1-x}])$ generate a rank two free \mathbb{Z} -submodule of G . Further, there is a \mathbb{Z} -module automorphism of G that interchanges these two elements. In particular, $0 \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$ has unequal complements in G .*

We emphasize that G does *not* split off $\mathbb{Z} \oplus \mathbb{Z}$ as a direct summand by Corollary A.2.

Proof. Let $f : G \rightarrow (\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K$ be the isomorphism from Corollary 3.4. The first conclusion follows by Remark 3.1. Consider an involution of the ladder manifold $\mathbb{L}(S^n, S^n)$ (for example, a product of two reflections) that interchanges the stringers, reverses the orientation of each sphere S_j , and induces the involution ρ of $(\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K$ given by $[(r, \gamma, s)] \mapsto [(s, -\gamma, r)]$. The automorphism $\psi : G \rightarrow G$ given by $\psi = f^{-1} \circ \rho \circ f$ interchanges the desired elements. □

In our proof of the Main Theorem, we will need to algebraically detect the submodule $\mathbb{Z} \oplus 0$ of G . The previous corollary shows that this will require more of the end-cohomology algebra than just the top degree module. We will use base manifolds X with nontrivial cup products in order to algebraically detect this submodule.

4. Stringers, Surgered Stringers, and Ladders Based on Surfaces

This section classifies all stringers, surgered stringers, and ladder manifolds based on closed surfaces. It demonstrates various methods for distinguishing end-cohomology

algebras up to isomorphism. In interesting cases, the ring structure plays the deciding role. This classification of ladders based on surfaces answers a question raised by Calcut and Haggerty [3, p. 3295]. In addition, its proof is good preparation for the more complicated situations that arise in subsequent sections. Let Σ_g denote the closed, connected, and oriented surface of genus $g \in \mathbb{Z}_{\geq 0}$. Throughout this section, we use integer coefficients.

The end-cohomology algebra of the stringer $[0, \infty) \times \Sigma_g$ is

$$\tilde{H}_e^k([0, \infty) \times \Sigma_g) \cong \tilde{H}^k(\Sigma_g) \cong \begin{cases} \mathbb{Z} & \text{if } k = 2, \\ \mathbb{Z}^{2g} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The cup product pairing $H^1(\Sigma_g) \times H^1(\Sigma_g) \rightarrow \mathbb{Z}$ is nonsingular and is given by

$$\bigoplus_g \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The end-cohomology algebra of the surgered stringer $\mathbb{S}(\Sigma_g)$ is

$$\tilde{H}_e^k(\mathbb{S}(\Sigma_g)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[[\sigma]]/\mathbb{Z}[\sigma] & \text{if } k = 2, \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}[[\tau]]/\mathbb{Z}[\tau] & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The cup product is coordinatewise in the direct sum, vanishes in the second coordinate, and is that of the cohomology ring of Σ_g in the first coordinate.

Given $g_1, g_2 \in \mathbb{Z}_{\geq 0}$, the end-cohomology algebra of the ladder manifold $\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})$ is

$$\tilde{H}_e^k(\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})) \cong \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}[[\sigma]] \oplus \mathbb{Z})/K & \text{if } k = 2, \\ \mathbb{Z}^{2g_1} \oplus \mathbb{Z}[[\tau]]/\mathbb{Z}[\tau] \oplus \mathbb{Z}^{2g_2} & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $K = \{(\sum \beta_i, \beta, -\sum \beta_i) \mid \beta = \sum \beta_i \sigma^i \in \mathbb{Z}[\sigma]\} \cong \mathbb{Z}[\sigma]$. The cup product is coordinatewise in the direct sum and vanishes in the middle coordinate. Define the matrices

$$C = \begin{bmatrix} 0 & \llbracket(1, 0, 0)\rrbracket \\ \llbracket-(1, 0, 0)\rrbracket & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \llbracket(0, 0, 1)\rrbracket \\ \llbracket-(0, 0, 1)\rrbracket & 0 \end{bmatrix},$$

where $\llbracket\alpha\rrbracket$ is the class of α in $(\mathbb{Z} \oplus \mathbb{Z}[[\sigma]] \oplus \mathbb{Z})/K$. For degree one elements, the cup product in the first coordinate is given by $\bigoplus_{g_1} C$, and in the third coordinate by $\bigoplus_{g_2} D$.

Of course, all of these manifolds may be capped with compact 3-manifolds (handlebodies, for example) to eliminate boundary and obtain open, one-ended 3-manifolds. However, compact caps will not alter the isomorphism types of their

graded end-cohomology algebras (which is our focus). So, we choose to work with the non-capped manifolds. We will use the following basic fact.

Lemma 4.1. *Let F be a free \mathbb{Z} -module of finite rank. Let G and H be submodules of F . Then $\text{rank}(G \cap H) \geq \text{rank}(G) + \text{rank}(H) - \text{rank}(F)$.*

Proof. The hypotheses imply that G , H , and $G + H$ are free \mathbb{Z} -modules of rank at most $\text{rank}(F)$ [8, p. 460]. We have the exact sequence of free \mathbb{Z} -modules

$$0 \rightarrow G \cap H \rightarrow G \oplus H \rightarrow G + H \rightarrow 0,$$

where the second map is $g \mapsto (g, -g)$ and the third map is $(g, h) \mapsto g + h$. Recall two facts: (i) if E is a free \mathbb{Z} -module, then $\text{rank}(E) = \dim_{\mathbb{Q}}(E \otimes_{\mathbb{Z}} \mathbb{Q})$ [8, pp. 373, 471] and (ii) tensoring with \mathbb{Q} is an exact functor (since \mathbb{Q} is a flat \mathbb{Z} -module [8, p. 401]). It follows that

$$\text{rank}(G) + \text{rank}(H) = \text{rank}(G \cap H) + \text{rank}(G + H) \leq \text{rank}(G \cap H) + \text{rank}(F)$$

as desired. □

Now, we will classify up to isomorphism the algebras listed for the three types of manifolds: stringers, surgered stringers, and ladder manifolds based on surfaces. The classification of these manifolds up to various types of equivalence will then readily follow. Plainly, $\mathbb{L}(X, Y) \approx \mathbb{L}(Y, X)$ for any manifolds X and Y .

Theorem 4.2. *Two of the algebras listed are isomorphic if and only if their corresponding manifolds have the same type and are based on surfaces of equal genus, with the exception: for each $g \in \mathbb{Z}_{\geq 0}$ the algebras for $\mathbb{S}(\Sigma_g)$, $\mathbb{L}(\Sigma_g, \Sigma_0)$, and $\mathbb{L}(\Sigma_0, \Sigma_g)$ are isomorphic. In particular, the algebras for $\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})$ and $\mathbb{L}(\Sigma_{g_3}, \Sigma_{g_4})$ are isomorphic if and only if $\{g_1, g_2\} = \{g_3, g_4\}$.*

Proof. For stringers based on surfaces with unequal genus, the algebras are distinguished by the ranks of \tilde{H}_e^1 . The algebras for a stringer and a surgered stringer or a ladder manifold are distinguished by the cardinalities of \tilde{H}_e^1 . Corollary A.2 implies that the algebras for surgered stringers based on surfaces with unequal genus are distinguished by the ranks of the duals of \tilde{H}_e^1 .

For each $g \in \mathbb{Z}_{>0}$, the manifolds $\mathbb{S}(\Sigma_g)$ and $\mathbb{L}(\Sigma_g, \Sigma_0) \approx \mathbb{L}(\Sigma_0, \Sigma_g)$ have diffeomorphic ends by Proposition 3.2. So, their algebras are isomorphic. In all other cases, the algebras for $\mathbb{S}(\Sigma_g)$ and $\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})$ are not isomorphic. If $g_2 = 0$ and $g_1 \neq g$ (or $g_1 = 0$ and $g_2 \neq g$), then use the ranks of the duals of \tilde{H}_e^1 . If $g_1 \neq 0$ and $g_2 \neq 0$, then use the ranks of the (degree two) subgroups generated by all cup products of degree one elements. For $\mathbb{S}(\Sigma_g)$ this rank is zero or one, and for $\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})$ it is two (see Remark 3.1).

It remains to classify the algebras for ladder manifolds based on surfaces. Suppose the following is an isomorphism:

$$f : \tilde{H}_e^*(\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})) \rightarrow \tilde{H}_e^*(\mathbb{L}(\Sigma_{g_3}, \Sigma_{g_4})).$$

Corollary A.2 implies that the ranks of the duals of \tilde{H}_e^1 are $2g_1 + 2g_2$ and $2g_3 + 2g_4$, respectively. So, $g_1 + g_2 = g_3 + g_4$. Suppose, by way of contradiction, that $\{g_1, g_2\} \neq \{g_3, g_4\}$. Then $g_1 + g_2 = g_3 + g_4$ implies that some g_i is strictly greater than the other three. Without loss of generality, we have

$$g_1 > g_3 \geq g_4 > g_2 \geq 0.$$

We will reach a contradiction using the ring structures. First, we eliminate the $\mathbb{Z}[[\tau]]/\mathbb{Z}[\tau]$ summands in an isomorphism invariant manner. Let J denote the set of elements in $\tilde{H}_e^1(\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2}))$ that are sent to 0 by every element in the dual of $\tilde{H}_e^1(\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2}))$. Note that J is a subgroup of $\tilde{H}_e^1(\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2}))$ and, in fact, is an ideal in $\tilde{H}_e^*(\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2}))$. Similarly, we define the ideal J' in $\tilde{H}_e^*(\mathbb{L}(\Sigma_{g_2}, \Sigma_{g_3}))$. Evidently, $f(J) = J'$ and so we obtain an induced isomorphism of the quotient algebras where we mod out by J and J' , respectively. Corollary A.2 implies that $J = 0 \oplus \mathbb{Z}[[\tau]]/\mathbb{Z}[\tau] \oplus 0$ (and similarly for J'). Therefore, we have an isomorphism $\bar{f} : A \rightarrow B$ of the algebras

$$A = \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}[[\sigma]] \oplus \mathbb{Z})/K & \text{if } k = 2, \\ \mathbb{Z}^{2g_1} \oplus 0 \oplus \mathbb{Z}^{2g_2} & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases} \quad B = \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}[[\sigma]] \oplus \mathbb{Z})/K & \text{if } k = 2, \\ \mathbb{Z}^{2g_3} \oplus 0 \oplus \mathbb{Z}^{2g_4} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $V = \mathbb{Z}^{2g_1} \oplus 0 \oplus 0$, a rank $2g_1$ and degree one submodule of A . Recalling Remark 3.1, cup products of elements of V generate $(\mathbb{Z} \oplus 0 \oplus 0) + K$, a rank one and degree two submodule of A . We will show that cup products of elements of $\bar{f}(V)$ generate a rank *two* and degree two submodule of B . This contradiction will complete the proof.

Note the following facts. For each element $0 \neq \alpha \in V$, there exists $\alpha' \in V$ such that $\alpha \cup \alpha' \neq 0$ (since the degree one cup product pairing for Σ_{g_1} is nonsingular). As \bar{f} is an isomorphism, the previous fact holds for $\bar{f}(V)$ as well. If $\gamma \in \mathbb{Z}^{2g_3} \oplus 0 \oplus 0$ and δ has degree one, then $\gamma \cup \delta \in (\mathbb{Z} \oplus 0 \oplus 0) + K$. Similarly, if $\gamma \in 0 \oplus 0 \oplus \mathbb{Z}^{2g_4}$ and δ has degree one, then $\gamma \cup \delta \in (0 \oplus 0 \oplus \mathbb{Z}) + K$. The last two facts hold since the cup product is coordinatewise.

Recalling that $g_1 + g_2 = g_3 + g_4$ and $g_1 > g_3 \geq g_4 > g_2 \geq 0$, Lemma 4.1 implies that there exist elements

$$\begin{aligned} 0 \neq \alpha &\in \bar{f}(V) \cap (\mathbb{Z}^{2g_3} \oplus 0 \oplus 0), \\ 0 \neq \beta &\in \bar{f}(V) \cap (0 \oplus 0 \oplus \mathbb{Z}^{2g_4}). \end{aligned}$$

By the previous paragraph, there exist $\alpha', \beta' \in \bar{f}(V)$ such that

$$\begin{aligned} 0 \neq \alpha \cup \alpha' &\in (\mathbb{Z} \oplus 0 \oplus 0) + K, \\ 0 \neq \beta \cup \beta' &\in (0 \oplus 0 \oplus \mathbb{Z}) + K. \end{aligned}$$

By Remark 3.1, $(\mathbb{Z} \oplus 0 \oplus \mathbb{Z}) + K$ is free of rank two. So, these two nonzero cup products generate a rank two submodule of degree two. This contradiction completes the proof. \square

Corollary 4.3. *Ladder manifolds $\mathbb{L}(\Sigma_{g_1}, \Sigma_{g_2})$ and $\mathbb{L}(\Sigma_{g_3}, \Sigma_{g_4})$ based on surfaces of genera $g_1, g_2, g_3, g_4 \in \mathbb{Z}_{\geq 0}$ are proper homotopy equivalent if and only if $\{g_1, g_2\} = \{g_3, g_4\}$. Hence, the same classification holds up to homeomorphism and up to diffeomorphism.*

5. End-Cohomology Algebra of Binary End-Sum

We present a proof of an unpublished result of Henry King. It computes the end-cohomology algebra of a binary end-sum in terms of the algebras of the two summands together with certain ray-fundamental classes determined by the rays used to perform the end-sum.

First, recall the analog for classical connected sum. Consider two closed, connected, oriented n -manifolds X and Y . The reduced cohomology ring $\tilde{H}^*(X \# Y)$ is isomorphic to the quotient of the sum $\tilde{H}^*(X) \oplus \tilde{H}^*(Y)$ by the principal ideal generated by $([X]^*, -[Y]^*)$ where $[X]^* \in H^n(X)$ and $[Y]^* \in H^n(Y)$ are the cohomology fundamental classes dual to the respective homology (orientation) fundamental classes. The cup product is coordinatewise in the sum. For the unreduced ring $H^*(X \# Y)$, let P be the subring of $H^*(X) \oplus H^*(Y)$ consisting of all elements of positive degree and only those of degree zero of the form (r, r) for $r \in \mathbb{Z}$. The desired ring is the quotient of P by the principal ideal generated by $([X]^*, -[Y]^*)$. One may prove these well-known facts by an argument structurally the same as our proof of Theorem 5.4. For end-sum and end-cohomology, the cohomology fundamental classes will be replaced by ray-fundamental classes that we now define.

Let M be a smooth, connected, oriented, noncompact manifold of dimension $n + 1 \geq 2$ with compact (possibly empty) boundary. Let $r \subset \text{Int } M$ be a ray, and let $\nu r \subset \text{Int } M$ be a smooth closed regular neighborhood of r in $\text{Int } M$ oriented as a codimension-zero submanifold of M . Orient the hyperplane $\partial \nu r \approx \mathbb{R}^n$ as the boundary of νr . We will define nonzero cohomology classes

$$[M, r]_e^* \in H_e^n(M, \nu r; R),$$

$$[r]_e^* \in \tilde{H}_e^n(M; R)$$

called (respectively) the *relative* and *absolute ray-fundamental classes* determined by the ray r . Our notation is chosen since, as will emerge, these elements are intimately related to classical fundamental classes of compact manifolds.

Recall that a Morse function $h: M \rightarrow \mathbb{R}$ is *exhaustive* provided h is proper and the image of h is bounded below.

Lemma 5.1. *There exists an exhaustive Morse function $h: M \rightarrow \mathbb{R}$ such that: (i) $h|_r$ is projection; (ii) $h|_{\nu r}$ has just one critical point, namely a global minimum*

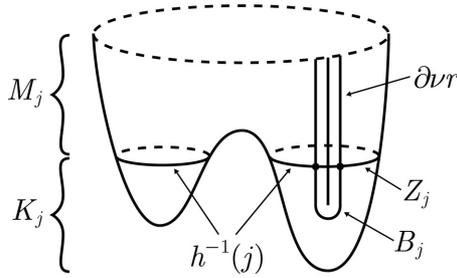


Fig. 8. Manifold M^{n+1} where the Morse function h is depicted as height. The $(n - 1)$ -sphere $\partial\widehat{Z}_j = \partial B_j$ is depicted as two dots.

in $\partial\nu r$; (iii) $h^{-1}([t, \infty)) \cap (\nu r, \partial\nu r) \approx [t, \infty) \times (D^n, S^{n-1})$ for each $t \geq 0$; and (iv) each $j \in \mathbb{Z}_{\geq 0}$ is a regular value of h .

Proof. By Whitney’s embedding theorem, we may assume $M \subset \mathbb{R}^{2n+3}$ is a submanifold that is embedded as a closed subspace. As $2n + 3 > 3$, we may assume, by an ambient isotopy of \mathbb{R}^{2n+3} , that r is straight in \mathbb{R}^{2n+3} . Next, ambiently untwist νr while fixing r . Define $h(x) := \|x - p\|^2 + c$ for an appropriate point $p \in \mathbb{R}^{2n+3}$ and $c \in \mathbb{R}$ [25, p. 36]. □

Let $h: M \rightarrow \mathbb{R}$ be a Morse function given by Lemma 5.1. For each $j \in \mathbb{Z}_{\geq 0}$, define $M_j := h^{-1}([j, \infty))$ and $K_j := h^{-1}((-\infty, j])$, both oriented as codimension-zero submanifolds of M (see Fig. 8). The K_j provide a compact exhaustion of M . For all sufficiently large j , the boundary of M (compact by hypothesis) is contained in the interior of K_j ; without loss of generality, we assume this holds for all $j \in \mathbb{Z}_{\geq 0}$ (shrink r toward infinity if necessary). So, for all $j \in \mathbb{Z}_{\geq 0}$, $h^{-1}(j) = K_j \cap M_j$ is a finite disjoint union of closed, connected n -manifolds. Let Z_j be the component of $h^{-1}(j)$ that meets νr , and let $\widehat{Z}_j := Z_j - \text{Int } \nu r$, both oriented as codimension-zero submanifolds of ∂K_j . The $(n - 1)$ -sphere $\partial\widehat{Z}_j$ is given the boundary orientation. Define $B_j := \partial\nu r \cap K_j \approx D^n$ oriented as a codimension-zero submanifold of $\partial\nu r$. Observe that $\partial\widehat{Z}_j = \partial B_j$ as oriented $(n - 1)$ -spheres.

For each $j \in \mathbb{Z}_{\geq 0}$, we define the following^d (see Fig. 9):

$$\begin{aligned} \widehat{M}_j &:= M_j - \text{Int } \nu r, \\ F_j &:= \nu r \cap M_j \approx [j, \infty) \times D^n, \\ \widehat{F}_j &:= \partial\nu r \cap M_j \approx [j, \infty) \times S^{n-1}, \\ \Delta_j &:= \nu r \cap Z_j \approx D^n. \end{aligned}$$

^dNotational mnemonic: intuitively \widehat{X} is a “nicely punctured” copy of X .

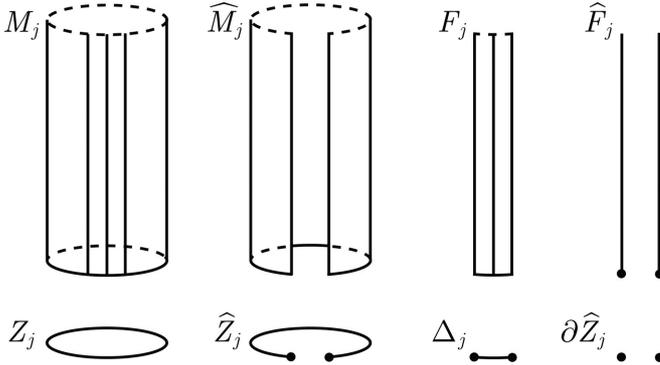


Fig. 9. Manifold M_j and some relevant submanifolds.

The fundamental class $[\partial\widehat{Z}_j]$ is our preferred generator of $H_{n-1}(\widehat{F}_j)$. By Universal Coefficients, its dual $[\partial\widehat{Z}_j]^*$ is our preferred generator of

$$\text{Hom}_{\mathbb{Z}}\left(H_{n-1}\left(\widehat{F}_j\right), R\right) \cong \widetilde{H}^{n-1}\left(\widehat{F}_j; R\right) \cong R,$$

where the latter isomorphism sends our preferred generator to $1 \in R$. In the direct system $\widetilde{H}^{n-1}(\widehat{F}_j; R)$, $j \in \mathbb{Z}_{\geq 0}$, each morphism is an isomorphism carrying one preferred generator to another. Therefore, the direct limit

$$\widetilde{H}_e^{n-1}(\partial\nu r; R) \cong \varinjlim \widetilde{H}^{n-1}(\widehat{F}_j; R) \cong R$$

has a preferred generator $[\partial\widehat{Z}_j]^*$ that is represented by each $[\partial\widehat{Z}_j]^*$. In the proof of Theorem 5.4, we use γ_M to denote $[\partial\widehat{Z}_j]^*$.

The inclusion $\iota_j: (\widehat{Z}_j, \partial\widehat{Z}_j) \rightarrow (\widehat{M}_j, \widehat{F}_j)$ induces the following diagram, where the rows are the long exact sequences for pairs:

$$\begin{array}{ccccccc} \longrightarrow & \widetilde{H}^{n-1}\left(\widehat{F}_j; R\right) & \xhookrightarrow{\delta_j} & H^n\left(\widehat{M}_j, \widehat{F}_j; R\right) & \longrightarrow & & \\ & \cong \downarrow \iota_j^* & & \downarrow \iota_j^* & & & \\ \longrightarrow & \widetilde{H}^{n-1}\left(\partial\widehat{Z}_j; R\right) & \xrightarrow[\cong]{\delta'_j} & H^n\left(\widehat{Z}_j, \partial\widehat{Z}_j; R\right) & \longrightarrow & & \end{array} \tag{5.1}$$

The diagram is commutative by naturality of the coboundary map. As δ'_j and the left ι_j^* are isomorphisms, δ_j is injective. We have the diagram

$$\begin{array}{ccccccc} H^{n-1}\left(\widehat{F}_j; R\right) & \xhookrightarrow{\delta_j} & H^n\left(\widehat{M}_j, \widehat{F}_j; R\right) & \xleftarrow[\cong]{\phi_j} & H^n\left(M_j, F_j; R\right) & \xrightarrow[\cong]{\psi_j} & H^n\left(M_j; R\right) \\ \left[\partial\widehat{Z}_j\right]^* & \longmapsto & \delta_j\left(\left[\partial\widehat{Z}_j\right]^*\right) & \longleftarrow & [M, r]_j^* & \longmapsto & [r]_j^* \end{array} \tag{5.2}$$

where ϕ_j is the excision isomorphism, ψ_j is the isomorphism from the long exact sequence for the pair, and $[M, r]_j^*$ and $[r]_j^*$ are defined by the diagram. Consider the

commutative diagram \mathcal{D} whose j th row, $j \in \mathbb{Z}_{\geq 0}$, equals (5.2). The four vertical maps in \mathcal{D} from row j to row $j + 1$ are inclusion induced. Passing to the direct limit in \mathcal{D} yields

$$\begin{CD} \widetilde{H}_e^{n-1}(\partial\nu r; R) @<\delta_M<< H_e^n(\widehat{M}, \partial\nu r; R) @<\phi_M<< \cong H_e^n(M, \nu r; R) @>\psi_M>> \widetilde{H}_e^n(M; R) \\ @VVV @VVV @VVV @VVV \\ [\partial\widehat{Z}_j]_e^* @>>> \delta_M([\partial\widehat{Z}_j]_e^*) @<<< [M, r]_e^* @>>> [r]_e^* \end{CD} \quad (5.3)$$

where δ_M is injective, $\widehat{M} := M - \text{Int } \nu r$, ϕ_M is the excision isomorphism, and ψ_M is the isomorphism from the long exact sequence for the pair. The relative and absolute ray-fundamental classes $[M, r]_e^* \in H_e^n(M, \nu r; R)$ and $[r]_e^* \in \widetilde{H}_e^n(M; R)$ are defined by (5.3).

Remark 5.2. (1) Let \mathcal{D}_e be the diagram \mathcal{D} augmented by the direct limit row (5.3) together with the canonical maps in each column from the terms in the direct system to the direct limit. The diagram \mathcal{D}_e is commutative and shows immediately that each $[M, r]_j^*$ and $[r]_j^*$ represent (respectively) $[M, r]_e^*$ and $[r]_e^*$. This observation holds *without* any additional assumptions on \mathcal{D} (such as surjectivity of the vertical maps in the last column).

(2) The ray-fundamental classes are well defined, up to isomorphism, independent of the choice of regular neighborhood νr by uniqueness of such neighborhoods. They are also independent of the Morse function h satisfying Lemma 5.1. To see this, let h' be another such Morse function and distinguish corresponding submanifolds of M by primes. As our Morse functions are exhaustive, each M_j contains M'_k for all sufficiently large k , and conversely. It follows that

$$H_e^*(M; R) \cong \varinjlim H^*(M_j; R) \cong \varinjlim H^*(M'_j; R)$$

and the latter of these isomorphisms carries the absolute ray-fundamental classes to one another. A similar argument applies to the relative case.

(3) If $r \subset M$ is neatly embedded, then we define the ray-fundamental classes $[M, r]_e^*$ and $[r]_e^*$ as follows. As in Fig. 10 (left), let $\tau r \subset M$ be a smooth closed tubular neighborhood of r in M (see Sec. 2 for our conventions on tubular neighborhoods). Let C be the boundary component of M containing ∂r . Let

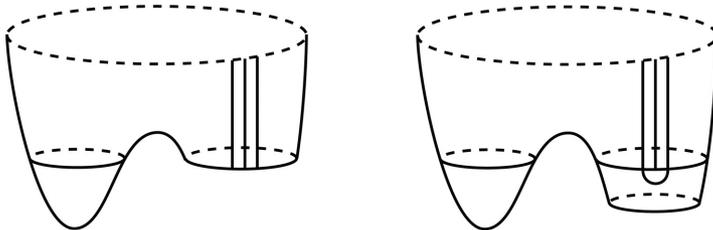


Fig. 10. Manifold M containing a neatly embedded ray r and a smooth closed tubular neighborhood τr (left), and $M' = M \cup$ (closed collar) (right).

M' equal M union a closed collar on C . The closed collar contains an $(n+1)$ -disk B such that $B \cup \tau r$ is a smooth closed regular neighborhood νr of r contained in the interior of M' as in Fig. 10 (right). Evidently

$$H_e^*(M', \nu r; R) \cong H_e^*(M, \tau r; R),$$

$$H_e^*(M'; R) \cong H_e^*(M; R).$$

We define the ray-fundamental classes $[M, r]_e^*$ and $[r]_e^*$ for $r \subset M$ to be the images under these isomorphisms of the ray-fundamental classes for $r \subset M'$.

- (4) The existence of the nonzero absolute ray-fundamental class $[r]_e^*$ implies that if M is a smooth, oriented, connected, noncompact manifold of dimension $n + 1 \geq 2$ with compact boundary, then $\tilde{H}_e^*(M; R)$ is nonzero. In particular, R injects into $H_e^n(M; R)$. For such a manifold M , $H_e^n(M; R)$ may indeed be the only nonzero cohomology group in $\tilde{H}_e^*(M; R)$. Consider the basic example of Euclidean space

$$\tilde{H}_e^*(\mathbb{R}^{n+1}; R) \cong \tilde{H}^*(S^n; R) \cong H^n(S^n; R) \cong R.$$

If M has noncompact boundary, then $\tilde{H}_e^*(M; R)$ may vanish. Consider the basic example of closed upper half-space \mathbb{R}_+^{n+1} which is proper homotopy equivalent to a ray

$$\tilde{H}_e^*(\mathbb{R}_+^{n+1}; R) \cong \tilde{H}_e^*([0, \infty); R) \cong 0.$$

Example 5.3. We will compute the absolute ray-fundamental class determined by a neat straight ray in a stringer. Fix a smooth, closed, connected, oriented n -manifold Z where $n \geq 1$. Let $\Delta \subset Z$ be a smoothly embedded n -disk, and let $z_0 \in \text{Int } \Delta$. So, $r = [0, \infty) \times \{z_0\}$ is a neat straight ray in the stringer $[0, \infty) \times Z$ with smooth closed tubular neighborhood $F = [0, \infty) \times \Delta$. We let $M_j = [j, \infty) \times Z$ and reuse the notation from Fig. 9 and thereafter.

We have the following diagram in integer homology:

$$\begin{array}{ccccccc}
 H_{n-1}(\partial \hat{Z}) & \xleftarrow[\cong]{\partial_*} & H_n(\hat{Z}, \partial \hat{Z}) & \xrightarrow[\cong]{\text{exc.}} & H_n(Z, \Delta) & \xleftarrow[\cong]{\text{l.e.}} & H_n(Z) \\
 [\partial \hat{Z}] & \longleftarrow & [\hat{Z}, \partial \hat{Z}] & \longrightarrow & [Z, \Delta] & \longleftarrow & [Z]
 \end{array} \tag{5.4}$$

Each of these groups is a copy of \mathbb{Z} . We claim that the preferred generators map as shown. It is well known that ∂_* is an isomorphism here. Seemingly less well known is the more explicit fact that $\partial_*([\hat{Z}, \partial \hat{Z}]) = [\partial \hat{Z}]$ for fundamental classes and the outward normal first orientation convention; a proof appears in Kreck [22, Theorem 8.1]. A moment of reflection reveals that the second and third isomorphisms in (5.4) send the preferred generators to the same generator, denoted $[Z, \Delta]$, of $H_n(Z, \Delta)$ as claimed.

The Universal Coefficients Theorem now yields the following since all relevant Ext groups vanish:

$$\begin{CD}
 H^{n-1}(\partial\widehat{Z}) @>\delta>> H^n(\widehat{Z}, \partial\widehat{Z}) @<\text{exc.}<< H^n(Z, \Delta) @>\text{l.e.}>> H^n(Z) \\
 @. @. @. @. \\
 [\partial\widehat{Z}]^* @>>> [\widehat{Z}, \partial\widehat{Z}]^* @<<< [Z, \Delta]^* @>>> [Z]^*
 \end{CD} \tag{5.5}$$

Diagram (5.5) is canonically isomorphic to row $j = 0$ in (5.2) by the obvious strong deformation retractions. The latter diagram is canonically isomorphic to the direct limit diagram (5.3) since every vertical map in \mathcal{D} is an isomorphism. Making the canonical identifications

$$H_e^n([0, \infty) \times Z; R) \cong H^n([0, \infty) \times Z; R) \cong H^n(Z; R) \cong R$$

with the last given by $[Z]^* \mapsto 1$, we have that $[r]^*_e = 1 \in R$. This completes our example.

Let (M, r) and (N, s) be end-sum pairs (see § 2.2) where M and N have the same dimension $n + 1 \geq 2$ and have compact (possibly empty) boundaries. Let $\nu r \subset \text{Int } M$ and $\nu s \subset \text{Int } N$ be smooth closed regular neighborhoods of r and s , respectively. Let $H \subset (M, r) \natural (N, s)$ denote the image of $\partial\nu r$ (which also equals the image of $\partial\nu s$). Let $u \subset H$ be an unknotted ray, and let νu be a smooth closed regular neighborhood of u in the interior of $S := (M, r) \natural (N, s)$.

Theorem 5.4 (H. King). *There are isomorphisms of graded R -algebras*

$$\begin{aligned}
 H_e^*(S, \nu u; R) &\cong (H_e^*(M, \nu r; R) \oplus H_e^*(N, \nu s; R)) / \langle ([M, r]^*_e, -[N, s]^*_e) \rangle, \\
 \widetilde{H}_e^*(S; R) &\cong (\widetilde{H}_e^*(M; R) \oplus \widetilde{H}_e^*(N; R)) / \langle ([r]^*_e, -[s]^*_e) \rangle,
 \end{aligned}$$

where $\langle ([M, r]^*_e, -[N, s]^*_e) \rangle$ and $\langle ([r]^*_e, -[s]^*_e) \rangle$ are homogeneous principal ideals of degree n .

Proof. Recall that S is obtained from the disjoint union of $\widehat{M} := M - \text{Int } \nu r$ and $\widehat{N} := N - \text{Int } \nu s$ by identifying $\partial\nu r$ and $\partial\nu s$ using an orientation reversing diffeomorphism. We have inclusions

$$\begin{aligned}
 i_M &: (\widehat{M}, \partial\nu r) \hookrightarrow (S, H), \\
 i_N &: (\widehat{N}, \partial\nu s) \hookrightarrow (S, H).
 \end{aligned}$$

We orient H so that $i_M| : \partial\nu r \rightarrow H$ is an orientation preserving diffeomorphism; it follows that $i_N| : \partial\nu s \rightarrow H$ is an orientation reversing diffeomorphism. Let $\omega \in H_e^{n-1}(H; R) \cong R$ be the preferred generator for this orientation. Hence, the

following hold for our preferred generators:

$$\begin{array}{ccc}
 \widetilde{H}_e^{n-1}(H; R) \xrightarrow[i_M|*]{\cong} \widetilde{H}_e^{n-1}(\partial\nu r; R) & & \widetilde{H}_e^{n-1}(H; R) \xrightarrow[i_N|*]{\cong} \widetilde{H}_e^{n-1}(\partial\nu s; R) \\
 \omega \longmapsto \longrightarrow \gamma_M & & \omega \longmapsto \longrightarrow -\gamma_N
 \end{array} \tag{5.6}$$

The long exact sequences for the pairs give isomorphisms

$$\begin{aligned}
 \psi_M &: H_e^*(M, \nu r; R) \xrightarrow{\cong} \widetilde{H}_e^*(M; R), \\
 \psi_N &: H_e^*(N, \nu s; R) \xrightarrow{\cong} \widetilde{H}_e^*(N; R), \\
 \psi_S &: H_e^*(S, \nu u; R) \xrightarrow{\cong} \widetilde{H}_e^*(S; R).
 \end{aligned}$$

Equation (5.3) shows that $\psi_M([M, r]_e^*) = [r]_e^*$ and $\psi_N([N, s]_e^*) = [s]_e^*$. So, the reduced cohomology result will follow immediately from the relative cohomology result. Further, the isomorphism ψ_S shows that it suffices to prove the following:

$$\widetilde{H}_e^*(S; R) \cong (H_e^*(M, \nu r; R) \oplus H_e^*(N, \nu s; R)) / \langle ([M, r]_e^*, -[N, s]_e^*) \rangle.$$

Consider the long exact sequence for the pair

$$\rightarrow \widetilde{H}_e^{k-1}(H; R) \xrightarrow{\delta} H_e^k(S, H; R) \xrightarrow{j^*} \widetilde{H}_e^k(S; R) \rightarrow \widetilde{H}_e^k(H; R) \rightarrow . \tag{5.7}$$

We claim that j^* is an isomorphism unless $k = n$, in which case j^* is surjective. As $\widetilde{H}_e^k(H; R) = 0$ for $k \neq n - 1$, the claim is clear except for surjectivity of j^* for $k = n - 1$. By exactness, it suffices to prove that

$$\delta_S: \widetilde{H}_e^{n-1}(H; R) \rightarrow H_e^n(S, H; R)$$

is injective. The inclusions i_M and i_N together with naturality of the coboundary map imply the following:

$$i_M^* \circ \delta_S = \delta_M \circ i_M|^*, \quad i_N^* \circ \delta_S = \delta_N \circ i_N|^*. \tag{5.8}$$

Either of these equations imply that δ_S is injective since both δ_M and δ_N are injective (see (5.3)) and both $i_M|^*$ and $i_N|^*$ are isomorphism. The claim is proved.

The claim implies that $\widetilde{H}_e^*(S; R)$ is isomorphic to the quotient of $H_e^*(S, H; R)$ by the kernel of j^* . By exactness of (5.7), this kernel is generated by $\delta_S(\omega)$.

By Corollary 2.8, the inclusions i_M and i_N induce the isomorphism

$$h: H_e^*(S, H; R) \xrightarrow{\cong} H_e^*(\widehat{M}, \partial\nu r; R) \oplus H_e^*(\widehat{N}, \partial\nu s; R),$$

where $h(\alpha) = (i_M^*(\alpha), i_N^*(\alpha))$. We also have the excision isomorphisms

$$\begin{aligned}
 \phi_M &: H_e^*(M, \nu r; R) \xrightarrow{\cong} H_e^*(\widehat{M}, \partial\nu r; R), \\
 \phi_N &: H_e^*(N, \nu s; R) \xrightarrow{\cong} H_e^*(\widehat{N}, \partial\nu s; R).
 \end{aligned}$$

Therefore, the theorem will follow provided we show that the image of $\delta_S(\omega)$ under the isomorphism h equals the image of $([M, r]_e^*, -[N, s]_e^*)$ under the isomorphism

$\phi_M \oplus \phi_N$. We have

$$\begin{aligned} h(\delta_S(\omega)) &= (i_M^*(\delta_S(\omega)), i_N^*(\delta_S(\omega))) \\ &= (\delta_M \circ i_M|_*(\omega), \delta_N \circ i_N|_*(\omega)) \\ &= (\delta_M(\gamma_M), \delta_N(-\gamma_N)) \\ &= (\phi_M([M, r]_e^*), \phi_N(-[N, s]_e^*)), \end{aligned}$$

where we used (5.8), (5.6), and (5.3). This completes our proof of the theorem. \square

Remark 5.5. (1) Recall that the number of ends of a space Y equals the rank of $H_e^0(Y; R)$ where R is a principal ideal domain [12, Proposition 13.4.11]. Thus, the reduced end-cohomology result in Theorem 5.4 implies that the number of ends (finite or infinite) of the binary end-sum S equals the sum of the numbers of ends of M and N minus one. In particular, if M and N are one-ended, then so is S .

(2) The results in this section likely hold in the piecewise-linear and topological categories and also for nonorientable manifolds with $R = \mathbb{Z}_2$. In this paper, we will not need these generalizations.

6. Ray-Fundamental Classes

6.1. Ray-fundamental classes in ladders

Fix X and Y to be closed, connected, oriented n -manifolds where $n \geq 2$. Let $\mathbb{L} := \mathbb{L}(X, Y)$ be the ladder manifold based on X and Y as defined in Sec. 3. Let r be a ray in \mathbb{L} emanating from $x_0 \in X_0$ and intersecting each S_j transversely as in Fig. 11. Let F be a smooth closed tubular neighborhood of r with a parameterization $\tau : [0, \infty) \times D^n \rightarrow F$ such that $r = \tau([0, \infty) \times \bar{0})$ and, for each j , $F \cap S_j = \tau(P_j \times D^n)$, where P_j is the (finite) set of preimages of points where r intersects S_j . If $p \in P_j$,

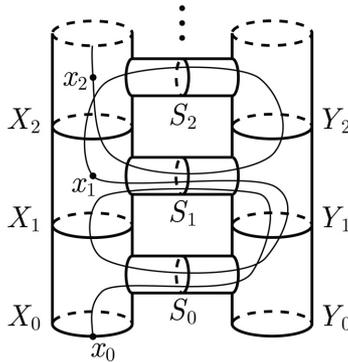


Fig. 11. Ray r in ladder manifold $\mathbb{L} = \mathbb{L}(X, Y)$.

then let D_p denote $\tau(p \times D^n)$, and let $D_0 = \tau(0 \times D^n) = F \cap X_0$. Viewing r as a properly embedded oriented submanifold of \mathbb{L} we may consider the \mathbb{Z} -intersection numbers $\varepsilon_{\mathbb{Z}}(r, S_j)$ (see [29, p. 68] or [16, p. 112]). Under this convention, $p \in P_j$ contributes $+1$ to $\varepsilon_{\mathbb{Z}}(r, S_j)$ if r passes from the X -side of \mathbb{L} to the Y -side on a small neighborhood of p and it contributes -1 if the reverse is true. Equivalently, give D_0 the orientation induced by X_0 and slide that orientation along the product structure of F to orient each D_p . Then $p \in P_j$ contributes $+1$ to $\varepsilon_{\mathbb{Z}}(r, S_j)$ if $D_p \hookrightarrow S_j$ is orientation preserving and -1 if $D_p \hookrightarrow S_j$ is orientation reversing.

Let $\widehat{\mathbb{L}} = \mathbb{L} - F^\circ$ and $\widehat{F} = F - F^\circ = \tau([0, \infty) \times S^{n-1})$, where F° denotes the topological interior of F as a subspace of \mathbb{L} . Our first goal is to understand the coboundary map $\delta : H^{n-1}(\widehat{F}) \rightarrow H^n(\widehat{\mathbb{L}}, \widehat{F})$. To accomplish this, we use the familiar diagram

$$H_{n-1}(\widehat{F}) \xleftarrow{\partial_*} H_n(\widehat{\mathbb{L}}, \widehat{F}) \xrightarrow[\cong]{\text{exc.}} H_n(\mathbb{L}, F) \xleftarrow[\cong]{\text{l.e.}} H_n(\mathbb{L}) \tag{6.1}$$

and examine the boundary map ∂_* .

By calculations in Sec. 3, the fundamental classes $[X_0]$, $[Y_0]$, and $[S_j]$, $j \in \mathbb{Z}_{\geq 0}$, form a free basis for $H_n(\mathbb{L})$. By the long exact sequence for (\mathbb{L}, F) and excision, $H_n(\widehat{\mathbb{L}}, \widehat{F})$ has a free basis consisting of the relative fundamental classes $[\widehat{X}_0, \partial\widehat{X}_0]$ of $\widehat{X}_0 := X_0 - \text{Int } D_0$ and $[\widehat{S}_j, \partial\widehat{S}_j]$ of the $\widehat{S}_j := S_j - \cup_{p \in P_j} \text{Int } D_p$ together with the fundamental class $[Y_0]$ of Y_0 . The $(n-1)$ -sphere $\partial\widehat{X}_0$ is given the boundary orientation; the fundamental class $[\partial\widehat{X}_0]$ is our preferred generator of $H_{n-1}(\widehat{F}) \cong \mathbb{Z}$. (This agrees with our orientation conventions in Sec. 5 where $\partial\widehat{Z}_0$ played the role of $\partial\widehat{X}_0$.) The orientation conventions established earlier in this section imply that $[\partial D_p] = -[\partial\widehat{X}_0]$ in $H_{n-1}(\widehat{F})$.

Now, $\partial_* : H_n(\widehat{\mathbb{L}}, \widehat{F}) \rightarrow H_{n-1}(\widehat{F})$ is determined by its action on this basis. We have $\partial_*([Y_0]) = 0$ and $\partial_*([\widehat{X}_0, \partial\widehat{X}_0]) = [\partial\widehat{X}_0]$ (see Example 5.3). For each $j \in \mathbb{Z}_{\geq 0}$, we have

$$\partial_*([\widehat{S}_j, \partial\widehat{S}_j]) = [\partial\widehat{S}_j] = \sum_{p \in P_j} -[\partial D_p] = \varepsilon_{\mathbb{Z}}(r, S_j) \cdot [\partial\widehat{X}_0].$$

We now return to the pertinent coboundary map δ where we will employ the following diagram:

$$\begin{array}{ccc} H^{n-1}(\widehat{F}) & \xrightarrow{\delta} & H^n(\widehat{\mathbb{L}}, \widehat{F}) \\ \cong \downarrow h' & & \cong \downarrow h \\ \text{Hom}_{\mathbb{Z}}(H_{n-1}(\widehat{F}), \mathbb{Z}) & \xrightarrow{\partial^*} & \text{Hom}_{\mathbb{Z}}(H_n(\widehat{\mathbb{L}}, \widehat{F}), \mathbb{Z}) \end{array} \tag{6.2}$$

Here, h' and h are the surjective homomorphisms provided by Universal Coefficients, and commutativity is verified in [17, p. 200]. Injectivity of h' and h requires

some specifics of the situation at hand, but both are immediate by Universal Coefficients when $H_{n-2}(\widehat{F})$ and $H_{n-1}(\widehat{\mathbb{L}}, \widehat{F})$ are torsion-free. That is clearly the case for $H_{n-2}(\widehat{F})$. Next, excision and the long exact sequence for (\mathbb{L}, F) imply that $H_{n-1}(\widehat{\mathbb{L}}, \widehat{F}) \cong H_{n-1}(\mathbb{L})$. By calculations in Sec. 3, the latter is isomorphic to $H_{n-1}(X_0) \oplus H_{n-1}(Y_0)$. By Poincaré duality, $H_{n-1}(X_0) \cong H^1(X_0)$ and similarly for Y_0 . By Universal Coefficients, degree one \mathbb{Z} -cohomology is always torsion-free and our assertion follows.

The Universal Coefficients Theorem gives the following diagram dual to (6.1) since all relevant Ext groups vanish:

$$H^{n-1}(\widehat{F}) \xrightarrow{\delta} H^n(\widehat{\mathbb{L}}, \widehat{F}) \xleftarrow[\cong]{\text{exc.}} H^n(\mathbb{L}, F) \xrightarrow[\cong]{\text{l.e.}} H^n(\mathbb{L}). \tag{6.3}$$

As in Sec. 3, we identify $H^n(\mathbb{L})$ with $\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z}$ where the dual fundamental class $[X_0]^*$ corresponds to the positive generator of the first summand, $[S_j]^*$ corresponds to the monomial x^j , and $[Y_0]^*$ corresponds to the positive generator in the third summand. We also identify $H^{n-1}(\widehat{F})$ with \mathbb{Z} by $[\partial\widehat{X}_0]^* \leftrightarrow 1$. Thus, the composite map $H^{n-1}(\widehat{F}) \rightarrow H^n(\mathbb{L})$ may be written as

$$\mu : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z}.$$

Define $\varepsilon_i = \varepsilon_{\mathbb{Z}}(r, S_i)$. With these conventions, diagram (6.2) and our description of ∂_* imply that $\mu(1) = (1, \sum_{i=0}^{\infty} \varepsilon_i x^i, 0)$.

By the end of Sec. 2.3, we have the canonical surjection

$$q : H^n(\mathbb{L}) \twoheadrightarrow H_e^n(\mathbb{L}) \cong (\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K.$$

By Remark 5.2(1), the following is now immediate.

Proposition 6.1. *Let r be a ray in \mathbb{L} emanating from $x_0 \in X_0$ and intersecting each S_i transversely, and let $\varepsilon_i = \varepsilon_{\mathbb{Z}}(r, S_i)$. Then the absolute ray-fundamental class determined by r is*

$$[r]_e^* = \llbracket (1, \sum_{i=0}^{\infty} \varepsilon_i x^i, 0) \rrbracket \in (\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K \cong H_e^n(\mathbb{L}). \tag{6.4}$$

Next, we prove a simple realization theorem whose proof is reminiscent of a Mazur–Eilenberg infinite swindle.

Proposition 6.2. *If $\alpha = \sum_{i=0}^{\infty} a_i x^i \in \mathbb{Z}[[x]]$, then there exists a ray r in \mathbb{L} emanating from $x_0 \in X_0$ such that $[r]_e^* = \llbracket (1, \alpha, 0) \rrbracket$.*

Proof. Recall the definition of $\mathbb{L}_{[j,k]} \subseteq \mathbb{L}$ in Sec. 3. Let $x_0 = (0, x) \in X_0$ be our usual basepoint, and for each $i \in \mathbb{Z}_{>0}$ choose $x_i = (i + 1/2, x) \in \mathbb{L}_{[i, i+1]}$ as in Fig. 11. Let $r_0 : [0, 1] \rightarrow \mathbb{L}_{[0,2]}$ be a smooth oriented path beginning at x_0 , ending at x_1 , and circling through the rungs of $\mathbb{L}_{[0,2]}$ so as to realize intersection numbers $\varepsilon_{\mathbb{Z}}(r_0, S_0) = a_0$ and $\varepsilon_{\mathbb{Z}}(r_0, S_1) = -a_0$. With respect to Fig. 11, this path will circle counterclockwise if $a_0 > 0$ and clockwise if $a_0 < 0$; if $a_0 = 0$, then it is a vertical arc.

Similarly, let $r_1 : [1, 2] \rightarrow \mathbb{L}_{[1,3]}$ be a path beginning at x_1 , ending at x_2 , and circling through the rungs of $\mathbb{L}_{[1,3]}$ so as to realize intersection numbers $\varepsilon_{\mathbb{Z}}(r_1, S_1) = a_0 + a_1$ and $\varepsilon_{\mathbb{Z}}(r_0, S_2) = -(a_0 + a_1)$. Notice that $\varepsilon_{\mathbb{Z}}(r_0 \cup r_1, S_0) = a_0$ and $\varepsilon_{\mathbb{Z}}(r_0 \cup r_1, S_1) = -a_0 + (a_0 + a_1) = a_1$.

In general, choose $r_k : [k, k + 1] \rightarrow \mathbb{L}_{[k,k+2]}$ beginning at x_k , ending at x_{k+1} , and realizing intersection numbers $\varepsilon_{\mathbb{Z}}(r_k, S_k) = \sum_{i=0}^k a_i$ and $\varepsilon_{\mathbb{Z}}(r_k, S_{k+1}) = -\sum_{i=0}^k a_i$. Then let $r : [0, \infty) \rightarrow \mathbb{L}$ be the union of these paths, adjusted, if necessary, to make it a smooth embedding. Choosing a nice smooth closed tubular neighborhood of r and applying the proof of Proposition 6.1 complete the proof. \square

6.2. Ray-fundamental classes in surgered stringers

The above propositions for ladders have simpler analogs for surgered stringers. Fix a closed oriented n -manifold X where $n \geq 2$. Let $\mathbb{S} := \mathbb{S}(X)$ be the surgered stringer based on X as defined in Sec. 3. Let r be a ray in \mathbb{S} emanating from $x_0 \in X_0$ and intersecting each S_j transversely. Recall the definition of $\mathbb{S}_{[j,k]}$ from Sec. 3. Working as we did in ladder manifolds, we consider the \mathbb{Z} -intersection numbers $\varepsilon_{\mathbb{Z}}(r, S_j)$. A point of $r \cap S_j$ at which r exits $\mathbb{S}_{[j,j+1/2]}$ contributes $+1$, and a point where r enters $\mathbb{S}_{[j,j+1/2]}$ contributes -1 .

As before, let F be a smooth closed tubular neighborhood of r chosen so there exists a parameterization $\tau : [0, \infty) \times D^n \rightarrow F$ with $r = \tau([0, \infty) \times \bar{0})$ and, for each j , $F \cap S_j = \tau(P_j \times D^n)$, where P_j is the set of preimages of $r \cap S_j$. Let $D_0 = F \cap X_0$ and for each $p \in P_j$ let $D_p = \tau(p \times D^n) \subseteq S_j$. Let $\widehat{\mathbb{S}} = \mathbb{S} - F^\circ$; $\widehat{F} = F - F^\circ$; $\widehat{X}_0 := X_0 - \text{Int } D_0$; and $\widehat{S}_j := S_j - \cup_{p \in P_j} \text{Int } D_p$. Using calculations from Sec. 3, the long exact sequence for $(\widehat{\mathbb{S}}, \widehat{F})$, excision, and notation as above for ladders, the relative fundamental classes $[\widehat{X}_0, \partial\widehat{X}_0]$ and $[\widehat{S}_j, \partial\widehat{S}_j]$, $j \in \mathbb{Z}_{\geq 0}$, form a free basis for $H_n(\widehat{\mathbb{S}}, \widehat{F})$. Our preferred generator of $H_{n-1}(\widehat{F}) \cong \mathbb{Z}$ is $[\partial\widehat{X}_0]$, and $[\partial D_p] = -[\partial\widehat{X}_0]$ in $H_{n-1}(\widehat{F})$.

The map $\partial_* : H_n(\widehat{\mathbb{S}}, \widehat{F}) \rightarrow H_{n-1}(\widehat{F})$ is given by $[\widehat{X}_0, \partial\widehat{X}_0] \mapsto [\partial\widehat{X}_0]$ and $[\widehat{S}_j, \partial\widehat{S}_j] \mapsto \varepsilon_{\mathbb{Z}}(r, S_j) \cdot [\partial\widehat{X}_0]$. The Universal Coefficients Theorem gives the following diagram:

$$H^{n-1}(\widehat{F}) \xrightarrow{\delta} H^n(\widehat{\mathbb{S}}, \widehat{F}) \xleftarrow[\cong]{\text{exc.}} H^n(\mathbb{S}, F) \xrightarrow[\cong]{\text{l.e.}} H^n(\mathbb{S}). \tag{6.5}$$

Identify $H^n(\mathbb{S})$ with $\mathbb{Z} \oplus \mathbb{Z}[[x]]$ by $[X_0]^* \leftrightarrow (1, 0)$ and $[S_j]^* \leftrightarrow (0, x^j)$. Identify $H^{n-1}(\widehat{F})$ with \mathbb{Z} by $[\partial\widehat{X}_0]^* \leftrightarrow 1$. The composite map $H^{n-1}(\widehat{F}) \rightarrow H^n(\mathbb{S})$ is now written as $\mu : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}[[x]]$. Define $\varepsilon_i = \varepsilon_{\mathbb{Z}}(r, S_i)$. Then $\mu(1) = (1, \sum_{i=0}^{\infty} \varepsilon_i x^i)$. We have the canonical surjection

$$q : H^n(\mathbb{S}) \rightarrow H_e^n(\mathbb{S}) \cong \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x].$$

Our work yields the following.

Proposition 6.3. *Let r be a ray in \mathbb{S} emanating from $x_0 \in X_0$ and intersecting each S_i transversely, and let $\varepsilon_i = \varepsilon_{\mathbb{Z}}(r, S_i)$. Then the absolute ray-fundamental class*

determined by r is

$$[r]_e^* = (1, \llbracket \sum_{i=0}^\infty \varepsilon_i x^i \rrbracket) \in \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x] \cong H_e^n(\mathbb{S}).$$

Furthermore, if $\alpha = \sum_{i=0}^\infty a_i x^i \in \mathbb{Z}[[x]]$, then there exists a ray r in \mathbb{S} emanating from $x_0 \in X_0$ such that $[r]_e^* = (1, \llbracket \alpha \rrbracket)$.

The proof of the realization result in this proposition is simpler than that for ladder manifolds. No “swindle” is needed here.

7. Proof of the Main Theorem

We first prove the Main Theorem using specific one-ended, open 4-manifolds. Then we describe various ways of adapting the proof to other one-ended, open manifolds. Let $T^k = \times_k S^1$ be the k -torus. Define

$$M = \mathbb{S}(T^3) \cup_\partial (T^2 \times D^2),$$

$$N = ([0, \infty) \times (S^1 \times S^2)) \cup_\partial (S^1 \times D^3).$$

So, M is the surgered stringer based on T^3 capped with $T^2 \times D^2$, and N is the stringer based on $S^1 \times S^2$ capped with $S^1 \times D^3$ as in Fig. 12.

Let $\alpha = \sum_{i=0}^\infty a_i x^i \in \mathbb{Z}[[x]]$. By Proposition 6.3, there is a ray $r \subset \text{Int } M$ such that

$$[r]_e^* = (1, \llbracket \alpha \rrbracket) \in \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x] \cong \tilde{H}_e^3(M).$$

As N is one-ended, collared at infinity, and has dimension at least four, it contains a unique ray up to ambient isotopy. So, let $s \subset \text{Int } N$ be a straight ray as in Fig. 12.

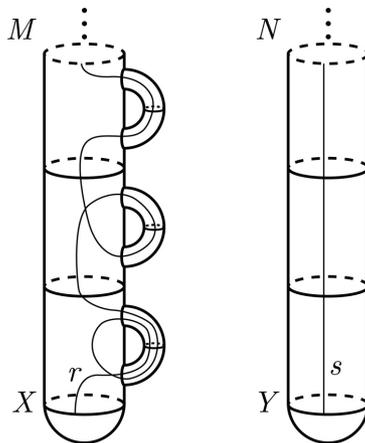


Fig. 12. Open manifolds M and N with rays $r \subset \text{Int } M$ and $s \subset \text{Int } N$.

By Example 5.3, we have

$$[s]_e^* = 1 \in \mathbb{Z} \cong \tilde{H}_e^3(N).$$

Let $S = (M, r) \natural (N, s)$. By Sec. 3 and Theorem 5.4, we have $\tilde{H}_e^*(S) \cong \mathcal{A}$ where

$$\tilde{H}_e^k(S) \cong \mathcal{A}_k := \begin{cases} ((\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]) \oplus \mathbb{Z})/I & \text{if } k = 3, \\ \mathbb{Z}^3 \oplus 0 \oplus \mathbb{Z} & \text{if } k = 2, \\ \mathbb{Z}^3 \oplus \mathbb{Z}[[\tau]]/\mathbb{Z}[\tau] \oplus \mathbb{Z} & \text{if } k = 1, \\ 0 \oplus 0 \oplus 0 & \text{otherwise.} \end{cases}$$

Here, I is the homogeneous ideal of degree 3 generated by $((1, [[\alpha]]), -1)$.

Next, let \mathcal{C} be an arbitrary graded \mathbb{Z} -algebra (possibly non-unital). We assign a sequence to \mathcal{C} by the following procedure:

- (1) Let $J \leq \mathcal{C}_1$ be the subgroup of elements in \mathcal{C}_1 that are sent to 0 by every element of the dual $\text{Hom}_{\mathbb{Z}}(\mathcal{C}_1, \mathbb{Z})$ of \mathcal{C}_1 . If J is not a two-sided ideal of \mathcal{C} , then return the empty sequence $()$ and end the procedure. Otherwise, J is a two-sided homogeneous ideal of \mathcal{C} .
- (2) Let $\mathcal{D} = \mathcal{C}/J$, which is a graded \mathbb{Z} -algebra.
- (3) In \mathcal{D} , let $U \leq \mathcal{D}_3$ be the subgroup generated by all products of three elements of degree one.
- (4) In \mathcal{D} , let $V \leq \mathcal{D}_3$ be the subgroup generated by elements that are a product of a degree one element and a degree two element but are not a product of three degree one elements.
- (5) If V is not infinite cyclic, then return the empty sequence $()$ and end the procedure. Otherwise, let v be either generator of V .
- (6) Let $\pi : \mathcal{D}_3 \rightarrow \mathcal{D}_3/U$ be the canonical homomorphism.
- (7) Return the height of $\pi(v)$ in \mathcal{D}_3/U and end the procedure.

The notion of the height of an element in an abelian group is reviewed below in Appendix A. Let $h(\mathcal{C})$ denote the sequence (empty or infinite in length) determined by the procedure.

Proposition 7.1. *If \mathcal{C} and \mathcal{C}' are isomorphic as graded \mathbb{Z} -algebras, then $h(\mathcal{C}) = h(\mathcal{C}')$. Applied to the specific case of the end-cohomology algebra of the end-sum S described above, this yields $h(\tilde{H}_e^*(S)) = h(\mathcal{A})$ which equals the height of $[[\alpha]] \in \mathbb{Z}[[x]]/\mathbb{Z}[x]$.*

Proof. Let $f : \mathcal{C} \rightarrow \mathcal{C}'$ be a graded \mathbb{Z} -algebra isomorphism. Note that f respects gradings, products, sums, and the \mathbb{Z} -module structure. In particular, f restricts to an isomorphism $f| : \mathcal{C}_1 \rightarrow \mathcal{C}'_1$, and the latter induces the isomorphism of dual modules $\text{Hom}_{\mathbb{Z}}(\mathcal{C}'_1, \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{C}_1, \mathbb{Z})$ given by $\psi \mapsto \psi \circ f|$. It follows that $f(J) = J'$. If J is not a two-sided ideal of \mathcal{C} , then J' is not a two-sided ideal of \mathcal{C}' and we have $h(\mathcal{C}) = () = h(\mathcal{C}')$. Otherwise, both J and J' are two-sided homogeneous ideals and f induces the graded \mathbb{Z} -algebra isomorphism $F : \mathcal{D} \rightarrow \mathcal{D}'$ given by

$F(x + J) = f(x) + J'$. As F respects gradings, products, sums, and the \mathbb{Z} -module structure, we get that F restricts to isomorphisms $\mathcal{D}_3 \rightarrow \mathcal{D}'_3$, $U \rightarrow U'$, and $V \rightarrow V'$. So, if V is not infinite cyclic, then neither is V' and we have $h(\mathcal{C}) = () = h(\mathcal{C}')$. Otherwise, V and V' are both infinite cyclic and $F(v) = \pm v'$. The isomorphism F induces the isomorphism $G : \mathcal{D}_3/U \rightarrow \mathcal{D}'_3/U'$ given by $G(x + U) = F(x) + U'$. So, the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{D}_3 & \xrightarrow[\cong]{F|} & \mathcal{D}'_3 \\
 \downarrow \pi & & \downarrow \pi' \\
 \mathcal{D}_3/U & \xrightarrow[\cong]{G} & \mathcal{D}'_3/U'
 \end{array} \tag{7.1}$$

In particular, G sends $\pi(v) \mapsto \pm \pi'(v')$. The sequence $h(\mathcal{C})$ is the height of $\pi(v) \in \mathcal{D}_3/U$, and the sequence $h(\mathcal{C}')$ is the height of $\pi'(v') \in \mathcal{D}'_3/U'$. These sequences are equal since height is invariant under isomorphism (see Corollary A.19) and sign change. This proves the first claim in the proposition and implies that $h(\widetilde{H}_e^*(S)) = h(\mathcal{A})$. It remains to show that $h(\mathcal{A})$ equals the height of $[\alpha] \in \mathbb{Z}[[x]]/\mathbb{Z}[x]$.

Applying the procedure to \mathcal{A} , Corollary A.2 implies that

$$J = 0 \oplus \mathbb{Z}[[\tau]]/\mathbb{Z}[\tau] \oplus 0 \leq \mathcal{A}_1.$$

Recall the product structure of \mathcal{A} given in Sec. 3. It implies that the product (in either order) of any element of J with any element of \mathcal{A} vanishes. So, J is a two-sided homogeneous ideal of \mathcal{A} . Taking the quotient of \mathcal{A} by J , we obtain the algebra \mathcal{D} where

$$\mathcal{D}_k := \begin{cases} ((\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]) \oplus \mathbb{Z})/I & \text{if } k = 3, \\
 \mathbb{Z}^3 \oplus 0 \oplus \mathbb{Z} & \text{if } k = 2, \\
 \mathbb{Z}^3 \oplus 0 \oplus \mathbb{Z} & \text{if } k = 1, \\
 0 \oplus 0 \oplus 0 & \text{otherwise.} \end{cases}$$

Define $U \leq \mathcal{D}_3$ and $V \leq \mathcal{D}_3$ as in the procedure. Using the well known \mathbb{Z} -cohomology rings of T^3 and $S^1 \times S^2$ (see [17, p. 216]), we have $U = ((\mathbb{Z} \oplus 0) \oplus 0)/I \cong \mathbb{Z}$ and $V = ((0 \oplus 0) \oplus \mathbb{Z})/I \cong \mathbb{Z}$ which are both infinite cyclic. Let v be either generator of V . Let $\pi : \mathcal{D}_3 \rightarrow \mathcal{D}_3/U$ be the canonical homomorphism. Then $h(\mathcal{A})$ equals the height of $\pi(v) \in \mathcal{D}_3/U$.

Conceptually, $\mathcal{A}_3 = \mathcal{D}_3$ is obtained from $\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$ by summing with \mathbb{Z} and then identifying the new \mathbb{Z} with an infinite cyclic subgroup of $\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$; essentially, this does not change the group. More precisely, consider the homomorphism $\eta : \mathcal{D}_3 \rightarrow \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]$ defined by $[(i, [\beta]), j] \mapsto (i + j, [\beta] + j[\alpha])$. Noting that $[(i, [\beta]), j] = [(i + j, [\beta] + j[\alpha]), 0]$, we see that η is an isomorphism of groups. Observe that $\eta(U) = \mathbb{Z} \oplus 0$, and $\eta(V)$ is the infinite cyclic subgroup generated by

(1, $[\alpha]$). We have the commutative diagram

$$\begin{array}{ccc}
 \mathcal{D}_3 & \xrightarrow[\cong]{\eta} & \mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x] \\
 \downarrow \pi & & \downarrow \\
 \mathcal{D}_3/U & \xrightarrow[\cong]{E} & (\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]) / (\mathbb{Z} \oplus 0) \xrightarrow[\cong]{} \mathbb{Z}[[x]]/\mathbb{Z}[x]
 \end{array} \tag{7.2}$$

where E is induced by η and the two downward homomorphisms in the triangle are the canonical projections which induce the horizontal isomorphism. At the level of elements, we have

$$\begin{array}{ccc}
 v & \longmapsto & \pm(1, [\alpha]) \\
 \downarrow & & \downarrow \\
 \pi(v) & \longmapsto & [\pm(1, [\alpha])] \longmapsto \pm[\alpha]
 \end{array} \tag{7.3}$$

As height is invariant under isomorphism and sign change, the height of $\pi(v) \in \mathcal{D}_3/U$ equals the height of $[\alpha] \in \mathbb{Z}[[x]]/\mathbb{Z}[x]$ proving the proposition. \square

As there exist uncountably many heights of elements in $\mathbb{Z}[[x]]/\mathbb{Z}[x]$ (see Lemma A.17), the Main Theorem is proved. Crucial to our proof was the detection of the specific subgroups U and V in an isomorphically invariant manner. To enable this, we chose manifolds with useful cup product structures. In the absence of such cup products, results at the end of Sec. 3 show that these subgroups cannot be so detected.

We close this section with a sample of variations of our proof of the Main Theorem. Always, we consider a pair of one-ended, open m -manifolds M and N .

(1) To prove the Main Theorem for each dimension $m \geq 5$, consider the manifolds

$$\begin{aligned}
 M &= \mathbb{S}(S^2 \times S^{m-3}) \cup_{\partial} (S^2 \times D^{m-2}), \\
 N &= ([0, \infty) \times (S^1 \times S^{m-2})) \cup_{\partial} (S^1 \times D^{m-1}).
 \end{aligned}$$

The proof is the same, except we consider the subgroups U and V of

$$\mathcal{D}_{m-1} \cong ((\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]) \oplus \mathbb{Z})/I,$$

where $U = ((\mathbb{Z} \oplus 0) \oplus 0)/I \cong \mathbb{Z}$ is the subgroup generated by elements that are a product of a degree two element and a degree $m - 3$ element (if $m = 5$, then this means the product of two degree two elements), and $V = ((0 \oplus 0) \oplus \mathbb{Z})/I \cong \mathbb{Z}$ is the subgroup generated by elements that are a product of a degree one element and a degree $m - 2$ element.

(2) For dimension $m = 3$, consider closed, oriented surfaces Σ_{g_1} and Σ_{g_2} of distinct positive genera $g_1 > g_2$ (the case $g_1 < g_2$ may be handled similarly). Consider

the manifolds

$$M = \mathbb{S}(\Sigma_{g_1}) \cup_{\partial} H_{g_1},$$

$$N = ([0, \infty) \times \Sigma_{g_2}) \cup_{\partial} H_{g_2},$$

where boundaries are capped with handlebodies. As in the main proof, we let $S = (M, r) \natural (N, s)$ and, by Sec. 4 and Theorem 5.4, we have

$$\mathcal{D}_k := \begin{cases} ((\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x]) \oplus \mathbb{Z})/I & \text{if } k = 2, \\ \mathbb{Z}^{2g_1} \oplus 0 \oplus \mathbb{Z}^{2g_2} & \text{if } k = 1, \\ 0 \oplus 0 \oplus 0 & \text{otherwise.} \end{cases}$$

In this proof, we assume $[[\alpha]] \neq 0$. So, products of degree one elements form a rank two subgroup of \mathcal{D}_2 ; products in the first factor generate $U := ((\mathbb{Z} \oplus 0) \oplus 0)/I$, and products in the third factor generate V which is the \mathbb{Z} -span of $[[(1, [[\alpha]], -1)]]$.

Consider subgroups $C \leq \mathcal{D}_1$ such that: (1) products of elements of C generate a rank one subgroup $D \leq \mathcal{D}_2$ and (2) the product of each element of C with each element of \mathcal{D}_1 lies in D . Note that $\mathbb{Z}^{2g_1} \oplus 0 \oplus 0$ is such a subgroup. Among all of these subgroups, consider one C' of maximal rank. Suppose that C' is not contained in $\mathbb{Z}^{2g_1} \oplus 0 \oplus 0$. Then using Lemma 4.1, we see that C' meets both $\mathbb{Z}^{2g_1} \oplus 0 \oplus 0$ and $0 \oplus 0 \oplus \mathbb{Z}^{2g_2}$ nontrivially. By Poincaré duality, elements that are the product of an element of C' and an element of \mathcal{D}_1 generate a rank two subgroup of \mathcal{D}_2 . This contradicts the defining properties of C' . Therefore, C' is contained in $\mathbb{Z}^{2g_1} \oplus 0 \oplus 0$. Among all such subgroups of maximal rank, $\mathbb{Z}^{2g_1} \oplus 0 \oplus 0$ is maximal with respect to containment. This algebraically distinguishes $\mathbb{Z}^{2g_1} \oplus 0 \oplus 0$ in an isomorphically invariant manner, and, hence, does the same for U . The rest of the proof is unchanged.

- (3) Similarly, one may prove the Main Theorem in each dimension $m \geq 3$ using ladder manifolds in place of surgered stringers. Details are left to the interested reader.
- (4) In all of the above manifolds used to prove the Main Theorem in some dimension m , one may use any compact caps to eliminate boundary and the proof is unchanged.

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Appendix A. Infinitely Generated Abelian Group Theory

We present some relevant results from the theory of infinitely generated abelian groups. This theory is subtle, beautiful, and (in our experience) not widely known

among topologists. For that reason, we provide proofs, as elementary as possible, for a few foundational results. These results are then applied to prove a few propositions tailored specifically to our needs in this paper. We close this appendix with a discussion of *height* in an abelian group.

A.1. Classical results

The additive abelian group $\mathbb{Z}[[x]] \cong \mathbb{Z} \times \mathbb{Z} \times \cdots$ is called the *Baer–Specker group*. Famously, it is not a free \mathbb{Z} -module; we include a proof of this fact below (see also [30]). The additive abelian group $\mathbb{Z}[x] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ is, of course, a free \mathbb{Z} -module with basis $\{1, x, x^2, \dots\}$. Throughout this and the next appendix, all maps are \mathbb{Z} -module homomorphisms. The *dual module* of a \mathbb{Z} -module M is the \mathbb{Z} -module

$$M^* := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}).$$

Fact A.1. If $f : \mathbb{Z}[[x]] \rightarrow \mathbb{Z}$ vanishes on $\mathbb{Z}[x]$, then $f = 0$.

Proof. Fix an integer $p > 1$. Consider the element

$$\alpha = a_0p^0x^0 + a_1p^1x^1 + a_2p^2x^2 + \cdots \in \mathbb{Z}[[x]],$$

where each $a_i \in \mathbb{Z}$. As f vanishes on $\mathbb{Z}[x]$, we get

$$f(\alpha) = f(a_kp^kx^k + a_{k+1}p^{k+1}x^{k+1} + \cdots) = p^k f(a_kp^0x^k + a_{k+1}p^1x^{k+1} + \cdots)$$

and so p^k divides $f(\alpha)$ for each integer $k > 0$. Hence, $f(\alpha) = 0$.

Next, fix coprime integers $p, q > 1$. Let $\gamma = \sum c_i x^i$ be an arbitrary element of $\mathbb{Z}[[x]]$. We wish to write $\gamma = \alpha + \beta$ where $\alpha = \sum a_i p^i x^i$ and $\beta = \sum b_i q^i x^i$. For each $i \geq 0$, we seek integers a_i and b_i such that $c_i = a_i p^i + b_i q^i$, which is always possible since p^i and q^i are coprime. Now, $f(\gamma) = f(\alpha) + f(\beta) = 0 + 0 = 0$. □

Corollary A.2. $(\mathbb{Z}[[x]]/\mathbb{Z}[x])^* = \{0\}$.

Proof. Let $f : \mathbb{Z}[[x]]/\mathbb{Z}[x] \rightarrow \mathbb{Z}$, and let $\pi : \mathbb{Z}[[x]] \twoheadrightarrow \mathbb{Z}[[x]]/\mathbb{Z}[x]$ be the canonical surjection. So, $f \circ \pi$ vanishes on $\mathbb{Z}[x]$. By Fact A.1, $f \circ \pi = 0$. As π is surjective, $f = 0$. □

Recall that $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ distributes over finite direct sums.

Corollary A.3. *The \mathbb{Z} -module $\mathbb{Z}[[x]]/\mathbb{Z}[x]$ is uncountable and torsion-free, but does not split off \mathbb{Z} as a direct summand and is not a free \mathbb{Z} -module.*

Proof. The first two claims are simple exercises. For the last two claims, otherwise $(\mathbb{Z}[[x]]/\mathbb{Z}[x])^*$ would be nonzero, contradicting Corollary A.2. □

Intuitively, $\mathbb{Z}[[x]]/\mathbb{Z}[x]$ is flexible and large regarding injective maps of free \mathbb{Z} -modules into it, but is rigid regarding maps to free \mathbb{Z} -modules.

Corollary A.4. $(\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x] \oplus \mathbb{Z})^* \cong \mathbb{Z}^2$.

Proof. Immediate by Corollary A.2. □

Fact A.1 also implies that two maps $\mathbb{Z}[[x]] \rightarrow \mathbb{Z}$ that agree on $\mathbb{Z}[x]$ must be equal (consider their difference). Combining this observation with projections, we see that two maps $\mathbb{Z}[[x]] \rightarrow \mathbb{Z}[[x]]$ that agree on $\mathbb{Z}[x]$ must be equal (see also [11, Lemma 94.1]). We mention that $\mathbb{Z}[x]^* \cong \mathbb{Z}[[x]]$ since $\mathbb{Z}[x]$ is free with \mathbb{Z} -basis $\{1, x, x^2, \dots\}$. The isomorphism is $f \mapsto \sum_{i \geq 0} f(x^i)x^i$. For completeness, we prove the “dual” fact that $\mathbb{Z}[[x]]^* \cong \mathbb{Z}[x]$.

The following fact is long known.

Fact A.5. If $f : \mathbb{Z}[[x]] \rightarrow \mathbb{Z}$, then $f(x^k) = 0$ for cofinitely many k .

Proof. Consider an element

$$\alpha = a_0! + a_1!x + a_2!x^2 + \dots \in \mathbb{Z}[[x]]$$

for integers $0 < a_0 < a_1 < a_2 < \dots$ to be determined. For each k , we have a tail of α denoted

$$a_k!\beta_k = a_k!x^k + a_{k+1}!x^{k+1} + \dots \in \mathbb{Z}[[x]],$$

where

$$\beta_k = \frac{a_k!}{a_k!}x^k + \frac{a_{k+1}!}{a_k!}x^{k+1} + \dots \in \mathbb{Z}[[x]].$$

Notice that β_k lies in $\mathbb{Z}[[x]]$ since the a_j ’s are increasing. We have

$$f(\alpha) = a_0!f(1) + a_1!f(x) + \dots + a_k!f(x^k) + a_{k+1}!f(\beta_{k+1}) \in \mathbb{Z}.$$

Hence

$$a_{k+1}!f(\beta_{k+1}) = f(\alpha) - a_0!f(1) - a_1!f(x) - \dots - a_k!f(x^k).$$

By the triangle inequality

$$a_{k+1}!|f(\beta_{k+1})| \leq |f(\alpha)| + a_0!|f(1)| + a_1!|f(x)| + \dots + a_k!|f(x^k)|.$$

Therefore,

$$|f(\beta_{k+1})| \leq \frac{|f(\alpha)|}{a_{k+1}!} + \frac{a_0!|f(1)| + a_1!|f(x)| + \dots + a_k!|f(x^k)|}{a_{k+1}!}.$$

The first term on the right side tends to zero as $k \rightarrow \infty$, and we may inductively choose the positive integers a_j so that the second term is less than $1/(k + 1)$. Hence, the non-negative integers $|f(\beta_{k+1})|$ tend to 0 as $k \rightarrow \infty$. So, $f(\beta_{k+1}) = 0$ for cofinitely many k .

Now, $a_k!\beta_k - a_{k+1}!\beta_{k+1} = a_k!x^k$ and so $f(x^k) = 0$ for cofinitely many k , as desired. □

Corollary A.6. $\mathbb{Z}[[x]]^* \cong \mathbb{Z}[x]$.

Proof. Let $f : \mathbb{Z}[[x]] \rightarrow \mathbb{Z}$. Fact A.5 implies that $f(x^k) = 0$ for cofinitely many k . Thus, $g : \mathbb{Z}[[x]] \rightarrow \mathbb{Z}$ defined by

$$g \left(\sum_{i \geq 0} c_i x^i \right) = \sum_{i \geq 0} c_i f(x^i) x^i$$

is a well-defined \mathbb{Z} -module homomorphism. As f and g agree on $\mathbb{Z}[x]$, we see that $f = g$. Note that the isomorphism $\mathbb{Z}[[x]]^* \rightarrow \mathbb{Z}[x]$ is $f \mapsto \sum_{i \geq 0} f(x^i) x^i$. \square

Corollary A.7. $\mathbb{Z}[[x]]$ is not a free \mathbb{Z} -module.

Proof. Otherwise, there is a \mathbb{Z} -basis for $\mathbb{Z}[[x]]$ which is necessarily uncountable and so $\mathbb{Z}[[x]]^*$ is uncountable. This contradicts Corollary A.6. \square

A.2. Applications of duals to end-cohomology of ladder manifolds

We now move to \mathbb{Z} -modules and algebras arising from ladder manifolds. Let $\mathbb{L}(X, Y)$ be a ladder manifold based on closed, connected, and oriented n -manifolds X and Y where $n \geq 2$. Recall from Sec. 3 that the degree n subgroup of the end-cohomology algebra of $\mathbb{L}(X, Y)$ is

$$\tilde{H}_e^n(\mathbb{L}(X, Y)) \cong (\mathbb{Z} \oplus \mathbb{Z}[[\sigma]] \oplus \mathbb{Z})/K,$$

where $K = \{(\sum \beta_i, \beta, -\sum \beta_i) \mid \beta = \sum \beta_i \sigma^i \in \mathbb{Z}[\sigma]\} \cong \mathbb{Z}[\sigma]$. Sometimes, we write x instead of σ in K . We begin by computing the dual module of $(\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K$.

Lemma A.8. If $h : \mathbb{Z}[[x]] \rightarrow \mathbb{Z}$ vanishes on

$$L := \left\{ \beta = \sum \beta_i x^i \in \mathbb{Z}[x] \mid \sum \beta_i = 0 \right\}$$

then $h = 0$.

Proof. By Fact A.5, there exists $k \geq 0$ such that $h(x^k) = 0$. Let $\alpha = \sum a_i x^i$ be an arbitrary element of $\mathbb{Z}[x]$ and let $n = \sum a_i$. Then $\alpha - nx^k \in L$ and $0 = h(\alpha - nx^k) = h(\alpha)$. So, h vanishes on $\mathbb{Z}[x]$. By Fact A.1, $h = 0$. \square

Corollary A.9. If $f : \mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z} \rightarrow \mathbb{Z}$ vanishes on K , then $f = 0$ on $0 \oplus \mathbb{Z}[[x]] \oplus 0$ and $f(r, \gamma, s) = j(r + s)$ for some fixed integer j .

Proof. Consider the inclusion $i : \mathbb{Z}[[x]] \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z}$ given by $i(\gamma) = (0, \gamma, 0)$. The composition $f \circ i : \mathbb{Z}[[x]] \rightarrow \mathbb{Z}$ vanishes on L . By Lemma A.8, $f \circ i = 0$. Thus, f vanishes on $0 \oplus \mathbb{Z}[[x]] \oplus 0$. As $(1, 1, -1) \in K$, we get

$$0 = f(1, 1, -1) = f(1, 0, 0) + f(0, 1, 0) - f(0, 0, 1) = f(1, 0, 0) - f(0, 0, 1)$$

and $f(1, 0, 0) = f(0, 0, 1)$. Define $j := f(1, 0, 0)$. Then $f(r, \gamma, s) = j(r + s)$. \square

Corollary A.10. $(\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K^* \cong \mathbb{Z}$.

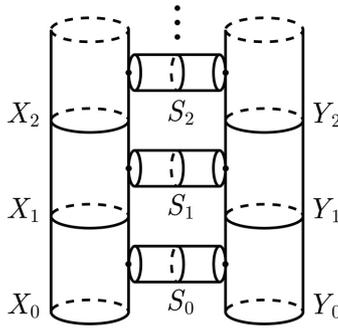


Fig. A.1. Wedge space $\mathbb{W}(X, Y)$.

Proof. Let $f : (\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K \rightarrow \mathbb{Z}$, and let $\pi : \mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z} \rightarrow (\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K$ be the canonical surjection. The composition $f \circ \pi$ vanishes on K . By the previous corollary, $f \circ \pi(r, \gamma, s) = j(r + s)$ for some fixed integer j . As π is surjective, $f(\llbracket(r, \gamma, s)\rrbracket) = j(r + s)$. In particular, $j = 1$ gives a generator for the dual module in question. \square

Corollary A.11. *The uncountable, torsion-free \mathbb{Z} -modules $\mathbb{Z} \oplus \mathbb{Z}[[x]]/\mathbb{Z}[x] \oplus \mathbb{Z}$ and $(\mathbb{Z} \oplus \mathbb{Z}[[x]] \oplus \mathbb{Z})/K$ are not isomorphic.*

Proof. They have nonisomorphic duals by Corollaries A.4 and A.10. \square

As an application of Corollary A.11, consider the space in Fig. A.1. The wedge space $\mathbb{W}(X, Y)$ based on X and Y (a nonmanifold) is obtained from the disjoint union of the stringers on X and Y by simply wedging on the rungs as shown. The end-cohomology algebra of $\mathbb{W}(X, Y)$ is

$$\tilde{H}_e^k(\mathbb{W}(X, Y); R) \cong \begin{cases} H^n(X; R) \oplus R[[\sigma]]/R[\sigma] \oplus H^n(Y; R) & \text{if } k = n, \\ H^k(X; R) \oplus 0 \oplus H^k(Y; R) & \text{if } 2 \leq k \leq n - 1, \\ H^1(X; R) \oplus R[[\tau]]/R[\tau] \oplus H^1(Y; R) & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the cup product is coordinatewise in the direct sum and vanishes in the middle coordinate. While the end-cohomology algebra of the wedge space $\mathbb{W}(X, Y)$ bears a striking resemblance to that of the ladder manifold $\mathbb{L}(X, Y)$, they are not isomorphic.

Corollary A.12. *The end-cohomology algebras of the ladder manifold $\mathbb{L}(X, Y)$ and the wedge space $\mathbb{W}(X, Y)$ are not isomorphic. In particular, these two spaces are not proper homotopy equivalent.*

Proof. The degree n subgroups of these two algebras are nonisomorphic by Corollary A.11. \square

A.3. Height in abelian groups

The notion of the height of an element plays an important role in the study of infinite abelian groups. Let G be an additive abelian group, $g \in G$, and $p > 1$ a prime integer. Consider the following equation in G for integers $k \geq 0$:

$$p^k x = g. \tag{†}$$

The height of $g \in G$ at p is

$$H_p(g) := k,$$

where $k \in \mathbb{Z}_{\geq 0}$ is maximal such that (†) has a solution $x \in G$. If (†) has a solution for every $k \in \mathbb{Z}_{\geq 0}$, then we write $H_p(g) = \infty$. Let $2 = p_1 < p_2 < \dots$ be the primes. The height of $g \in G$ is the sequence

$$H(g) := (H_{p_1}(g), H_{p_2}(g), \dots) \in \{0, 1, 2, \dots, \infty\}^{\mathbb{N}}. \tag{‡}$$

For example, $H(0) = (\infty, \infty, \infty, \dots)$ in every abelian group G . If G is a field of characteristic zero, then $H(g) = (\infty, \infty, \infty, \dots)$ for all $g \in G$. If G is a field of characteristic p and $0 \neq g \in G$, then $H_p(g) = 0$ and $H_q(g) = \infty$ for each prime $q \neq p$. In general, the height of g depends on G , since the solutions x of (†) are required to lie in G . For the sake of intuition, it is useful to look at some examples. In what follows, the partial ordering \preceq on $\{0, 1, 2, \dots, \infty\}^{\mathbb{N}}$ is defined by: $(m_i)_{i \in \mathbb{N}} \preceq (n_i)_{i \in \mathbb{N}}$ if and only if $m_i \leq n_i$ for all $i \in \mathbb{N}$.

Example A.13. Let $G = \mathbb{Z}$ and $g = 1400$. The prime factorization $1400 = 2^3 5^2 7$ reveals that $H_2(g) = 3$, $H_5(g) = 2$, $H_7(g) = 1$, and $H_p(g) = 0$ for all other primes p . Inserting this data into (‡) give us

$$H(1400) = (3, 0, 2, 1, 0, 0, 0, \dots).$$

More generally, if $0 \neq g = \prod_{i=1}^{\infty} p_i^{k_i}$, then

$$H(g) = (k_1, k_2, k_3, \dots),$$

where all $k_i < \infty$ and all but finitely many k_i are zero.

Example A.14. Let $G = \mathbb{Z}[x]$ and $\alpha = \sum_{i=0}^n a_i x^i \in \mathbb{Z}[x]$ where $a_n \neq 0$. Recall that the content of α is

$$c(\alpha) = \gcd(a_0, a_1, \dots, a_n) \in \mathbb{Z}_{>0}.$$

It is straightforward to verify that the height of $\alpha \in \mathbb{Z}[x]$ equals the height of $c(\alpha) \in \mathbb{Z}$. Thus, as in the previous example, all heights of nonzero elements contain finitely many nonzero entries each of which is finite.

Example A.15. Let $G = \mathbb{Z}[[x]]$ and $0 \neq \alpha = \sum_{i=0}^{\infty} a_i x^i \in \mathbb{Z}[[x]]$. We define the *content* of α to be

$$c(\alpha) = \gcd(a_0, a_1, a_2, \dots) \in \mathbb{Z}_{>0}.$$

This is well defined since some $a_n \neq 0$ and

$$\infty > \gcd(a_0, a_1, \dots, a_n) \geq \gcd(a_0, a_1, \dots, a_{n+1}) \geq \gcd(a_0, a_1, \dots, a_{n+2}) \geq \dots$$

Only finitely many of these inequalities can be strict by the well-ordering principle. In particular, $c(\alpha) = \gcd(a_0, a_1, \dots, a_N)$ for some non-negative integer $N = N(\alpha)$. Again, it is straightforward to verify that the height of $\alpha \in \mathbb{Z}[[x]]$ equals the height of $c(\alpha) \in \mathbb{Z}$, which yields exactly the same collection of heights as in the previous two examples. Therefore, even though $\mathbb{Z}[[x]]$ is uncountable, its elements realize only countably many heights.

Example A.16. In $G = \mathbb{Z}[[x]]/\mathbb{Z}[x]$ height becomes more interesting. Consider, for example

$$\begin{aligned} H(\llbracket 2 + 2x + 2x^2 + \dots \rrbracket) &= (1, 0, 0, \dots), \\ H(\llbracket 2^3 5^0 + 2^3 5^1 x + 2^3 5^2 x^2 + \dots \rrbracket) &= (3, 0, \infty, 0, 0, \dots), \\ H(\llbracket 2^1 x + 2^2 3^2 x^2 + 2^3 3^3 5^3 x^3 + \dots \rrbracket) &= (\infty, \infty, \infty, \dots). \end{aligned}$$

The key here is that, for any power series $\sum_{i=0}^{\infty} a_i x^i \in \mathbb{Z}[[x]]$ representing an element of $\mathbb{Z}[[x]]/\mathbb{Z}[x]$, we may ignore any finite initial sum $\sum_{i=0}^j a_i x^i$. This leads to the following.

Lemma A.17. *Let $G = \mathbb{Z}[[x]]/\mathbb{Z}[x]$ and let $h = (h_1, h_2, \dots) \in \{0, 1, 2, \dots, \infty\}^{\mathbb{N}}$ be a height. Then there exists $g \in G$ such that $H(g) = h$.*

Proof. The idea of the proof is contained in Example A.16. Let $2 = p_1 < p_2 < \dots$ be the rational primes. For each integer $i \geq 1$, define

$$a_i := p_1^{e(1,i)} p_2^{e(2,i)} \dots p_i^{e(i,i)} \in \mathbb{Z}_{>0},$$

where

$$e(n, i) := \begin{cases} i & \text{if } h_n \geq i, \\ h_n & \text{if } h_n < i. \end{cases}$$

Let $g = \llbracket \sum_{i=1}^{\infty} a_i x^i \rrbracket \in G$. Then the height of $g \in G$ is $H(g) = h$. □

For a general discussion of height, see [10, Chap. VII]. For our purposes, the crucial facts are Lemma A.17 and the following.

Lemma A.18. *Let $\phi : G \rightarrow G'$ be a homomorphism of abelian groups. Then for all $g \in G$ we have $H(g) \preceq H(\phi(g))$.*

Proof. Let $p > 1$ be a prime integer and suppose $g = p^k x$ in G . Then $\phi(g) = \phi(p^k x) = p^k \phi(x)$ in G' . Therefore, $H_p(g) \leq H_p(\phi(g))$. \square

Corollary A.19. *Let $\phi : G \rightarrow G'$ be an isomorphism of abelian groups. Then for all $g \in G$ we have $H(g) = H(\phi(g))$.*

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