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ARTIN PRESENTATIONS OF COMPLEX SURFACES

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ABSTRACT. We construct Artin presentations of infinitely many complex surfaces. Namely, for all elliptic surfaces $E(n)$, in particular for the Kummer surface $K3$. Thus, not only does AP theory contain an analogue of Donaldson's Theorem, but also a purely group-theoretic theory of Donaldson and Seiberg-Witten invariants.

Not surprisingly, our explicit Artin presentations for the Kummer surface are approachable with a computer using, say, MAGMA and provide a plethora of interesting examples pertaining to knot theory in $\mathbb{Z}$-homology 3-spheres.

1. Introduction

In the purely group-theoretic theory of Artin presentations, a smooth, compact, connected, simply-connected 4-manifold $W^4(r)$ with a connected boundary $\partial W^4(r) = M^3(r)$ is already determined, and can be reconstituted, from a certain presentation (an Artin presentation) of the fundamental group of its boundary [W1]. If the boundary is S^3 then of course the Artin presentation presents the trivial group. Even in this case the Artin presentation already encodes *all of the smooth structure* of the 4-manifold. Thus, it makes sense to ask whether an *arbitrary*, smooth, *closed*, connected, simply-connected 4-manifold is given by an Artin presentation.

We extend important work of Harer, Kas and Kirby [HKK] and show that all elliptic surfaces $E(n)$ admit Artin presentations. *This gives the first bridge between AP theory and algebraic geometry*. These Artin presentations are of special interest due to the fact that complex algebraic surfaces possess nontrivial Donaldson invariants. In particular, this augments the remarkable fact (Theorem 1 of [W1], [R] p. 621) that Donaldson's Theorem, despite being proved with gauge theory/connections (i.e. the smooth continuum), persists and survives the radical, discrete, purely group theoretic holography of AP theory.

The following illustrates the AP theory program concerning the computation of Seiberg-Witten and Donaldson invariants and shows that the group theoretic AP encoding goes much deeper than e.g. the mere encoding of a group through its presentation:

Recall González-Acuña's formula, [CS] p. 66, for the Rohlin invariant of a $\mathbb{Z}$-homology 3-sphere $\Sigma^3(r)$ given by an Artin presentation $r \in \mathcal{R}_n$ (for clarity we consider here only the case where $A(r)$ is the identity matrix, see section 2.1 for notation):

$$\mu(\Sigma^3(r)) = \frac{d^2-1}{8} \bmod 2,$$

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where $d = \triangle(-1)$, $\triangle$ being the Alexander polynomial of the associated presentation:

$$\langle x_1, \ldots, x_n \mid x_1 r_1 = r_1 x_2, x_2 r_2 = r_2 x_3, \ldots, x_{n-1} r_{n-1} = r_{n-1} x_n \rangle,$$

where the group obviously abelianizes to $\mathbb{Z}$.

This remarkable formula is entirely from the discrete theory of finitely presented groups: there is no need to mention cobordisms, spin structures, skein methods, Heegaard decompositions, representations into $SU(2)$, Riemannian metrics, infinite dimensional or moduli spaces, or indeed even the smooth continuum, nor do any metric dependence, wall crossing, or word problems arise here.

We remark that González-Acuña's formula already shows that an analogue of Floer theory should also appear in AP theory since the Rohlin invariant is the Euler characteristic (mod 2) in Floer theory. In fact, we suspect that 'the 8 of González-Acuña is the 8 of Floer'.

Concerning the importance of relating Donaldson and Floer theory, both mathematically and physically, see [D] p.63 and [Wi1] p.352.

Consider the more general problem concerning the relative Donaldson invariants [TB],[Wi1] of $W^4(r)$ which, when $A(r)$ is unimodular, take values in the Floer homology of $\partial W^4(r) = \Sigma^3(r)$.

The computational program of AP theory can be stated as: these invariants and others should be computed solely in function of the Artin presentation r in the discrete theory of finitely presented groups, just as, with González-Acuña's formula, this was done for the Rohlin invariant of $\Sigma^3(r)$.

This is entirely in the purely group-theoretic spirit of the Princeton School of Artin, Fox, Lyndon, Papakyriakopoulos, Stallings, et al. and extends their approach, as far as $3D/4D$ manifold theory is concerned, to its natural metamathematical boundary.

Immediate natural, important general questions arise (both mathematical and physical):

1. *Since AP theory dispenses not only with metrics but even topology, what becomes of Witten's celebrated Feynmanian formulation of Donaldson's invariants as correlation functions/expectation values* [D] *p. 53,* [Wi2], [Wi3], [AJ], [Di] *pp. 36, 39? What is the* topologically independent *(i.e. purely AP theoretical) analogue of Witten's* metric independent *Lagrangian for the Casson theory* [AJ] *p.121? What does González-Acuña's formula for the Rohlin invariant suggest? Is the mysterious question about the relationship between the Donaldson invariants of oppositely oriented* X^4 *related to the purely group-theoretic one of finding the inverse in* $\mathcal{R}_n$ *of an Artin presentation?*

2. *In the absence of moduli spaces, etc., is Witten's "mass-gap" discussion regarding Donaldson theory,* [Wi3] *pp.289-291, still relevant in AP theory?*

3. *Is the Denjoy-like inequivalence between Seiberg-Witten theory and Donaldson theory detectable in AP theory? Recall that Seiberg-Witten theory requires spinors and the Dirac operator, i.e. an underlying* C^1 *structure, whereas Donaldson's theory is valid on the wider class of Lipschitz manifolds* [D] *p.69,* [S], [DS].

4. *In general, the word problem obstructs the study of arbitrary smooth 4-manifolds. Although 4-manifolds in AP Theory are simply connected, we can still ask whether the group-theoretical physical questions of Geroch-Hartle* [GH] *(see also* [F]*) are still relevant when transferred to the group theory of 3-manifolds. Theorem I of* [W1] *seems to illustrate a purely group-theoretic Bohm-Aharonov phenomenon.*

5. *AP Theory does not just dispense with the smooth continuum, but also dispenses with integer (co)homology/intersection theory since all of this information is already given simply by the symmetric integer matrix $A(r)$. Hence, should e.g. the Kronheimer-Mrowka canonical basic class of $W^4(r)$,* when $\partial W^4(r) = S^3$, [D] p.52, [K], [St], *be already determined with Number Theory, à la Elkies* [E], [D] *p.67 and Borcherds* [B], *thus explaining the persistence of invariants constructed with the aid of a complex structure when this structure does not exist? For the same reason, difficult 'minimal genus' and 'simple type' problems,* [D] *p.68,* [St] *p.156, should be studied in this, their ultimate natural context, where artificial complications caused by the use of the smooth continuum are absent.*

It does not seem surprising, due to the basic nature of the $K3$ complex surface (e.g. it is the only 4D, closed, simply connected Calabi-Yau manifold and its quadratic form is the first even non-Donaldson form), that our Artin presentations lead to several interesting and instructive examples (section 3 ahead) which complement and extend to the 'softer' non-Donaldson case those examples obtained from such matrices as E_8, ϕ_{4n}, and the Coxeter-Todd extremal duodenary matrix $2D_{12}^2$ [W1].

2. The Artin presentations

The purpose of this section is to construct Artin presentations for all elliptic surfaces $E(n)$. This is carried out completely for $E(2)$, which is diffeomorphic to the Kummer surface $K3$ [GS], p.74, and follows mutatis mutandis for the others. The organization runs as follows: 2.1 is a brief discussion of Artin presentations and framed pure braids, in 2.2 we obtain a surgery diagram for $E(n)$ that is a framed pure braid, 2.3 provides an explicit algorithm (fixing all conventions) for obtaining an Artin presentation from a framed pure braid, and 2.4 combines everything obtaining the desired Artin presentation for $K3$.

(2.1) Artin presentations and pure braids. We begin by reviewing some of the fundamentals of AP theory. For a rigorous introduction to AP theory, proofs of the statements made below and a thorough bibliography we refer the reader to [W1].

Let $F_n = \langle x_1, \ldots, x_n \rangle$ be the free group on n-generators. An Artin presentation r is a balanced presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ satisfying the equation:

$$x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n), \quad \text{(AC)}$$

in F_n, which we will refer to as the Artin condition. The set of all Artin presentations on n-generators is denoted $\mathcal{R}_n$ and forms a group. By Ω_n we mean the compact 2-disk with n-holes and boundary $\partial\Omega_n$ equal to the disjoint union of $\partial_0, \partial_1, \ldots, \partial_n$ (see [W1] p.225). An Artin presentation $r \in \mathcal{R}_n$ determines, among other things, the following:

$\pi(r)$:	the group presented by r,
$M^3(r)$:	a closed orientable 3-manifold,
$W^4(r)$:	a smooth compact connected simply-connected 4-manifold,
$A(r)$:	an $n \times n$ symmetric integer matrix,
$h(r)$:	a self diffeomorphism of Ω_n unique up to isotopy fixing $\partial\Omega_n$ with $h\|_{\partial\Omega_n}$ equal to the identity.

The relationships between these objects are canonical. The manifold $M^3(r)$ bounds $W^4(r)$, has fundamental group isomorphic to $\pi(r)$, and is the open book defined by $h(r)$. The symmetric matrix $A(r)$ is the exponent sum matrix of r and also represents the intersection form of $W^4(r)$. The manifold $M^3(r)$ is a $\mathbb{Z}$-homology 3-sphere if and only if $\det A(r) = \pm 1$, and in this case we write $\Sigma^3(r)$ instead of $M^3(r)$.

An Artin presentation $r \in \mathcal{R}_n$ also determines an automorphism of F_n by the mapping $x_i \mapsto r_i^{-1} x_i r_i$. Namely, this is the automorphism $h_\# \colon \pi_1(\Omega_n, p_0) \to \pi_1(\Omega_n, p_0)$ where p_0 is a distinguished point in $\partial_0 \subset \partial\Omega_n$ and $x_1, \ldots, x_n$ represent the canonical generators (see Figure 9 ahead and [W1] p.225 and p.244). This view will prove useful when composing Artin presentations.

As pointed out in [W1], $\mathcal{R}_n$ is canonically isomorphic to $P_n \times \mathbb{Z}^n$, the framed pure braid group, where P_n is the pure braid group on n-strands. To see this, notice that $r \in \mathcal{R}_n$ determines $h = h(r)$ and h can be realized concretely in $\mathbb{R}^3$ by taking $\Omega_n \times I$ (I denotes the closed unit interval), suitably braiding the inner boundary tubes with one another, and twisting the inner boundary tubes by some integer numbers of complete revolutions (see [W1] p.245). Twisting the inner tubes can be accomplished by elementary Dehn twists about the ∂_i and these Dehn twists commute with all others. This braiding/twisting of the inner boundary tubes is easily seen to be equivalent to specifying both a pure braid (pure as $h|_{\partial\Omega_n} = id$) and an integer (the 'framing coefficient') for each strand.

Let $r \in \mathcal{R}_n$. The manifold $W^4(r)$ is defined in [W1] p. 250 as follows. Embed Ω_n in S^2 and extend h to all of S^2 by the identity. Then, extend this map to a self diffeomorphism of all of D^3, calling the result $H = H(r)$ (which is unique up to isotopy). Letting $W(H)$ be the mapping torus of H, $W^4(r)$ is defined to be $W(H)$ union $(n+1)$ 2-handles attached canonically. Notice that $W(H)$ is diffeomorphic to $D^3 \times S^1$ (= 0-handle $\cup$ 1-handle) as all orientation preserving self diffeomorphisms of D^3 are smoothly isotopic to the identity. We wish to examine this construction more closely. The self diffeomorphism h of Ω_n can be realized, as described in the previous paragraph, in $\mathbb{R}^3$ as $\Omega_n \times I$ with the inner boundary tubes braided and twisted; the map h of Ω_n is then obtained by bending the twisted $\Omega_n \times I$ around and sticking the ends $\Omega_n \times 0$ and $\Omega_n \times 1$ together in the canonical way, exactly as one does to close a braid. To construct H, one can first extend h to D^2 by taking the twisted $\Omega_n \times I$ and filling in the n

inner boundary tubes with n copies of $D^2 \times I$. One must take some care here. For each boundary tube $\partial_i \times I$, $i = 1, \ldots, n$, let p_i be a distinguished point (see Figure 9 ahead and [W1] p.225). Let $*$ be a distinguished point in ∂D^2. Then, when filling the i^{th} boundary tube $\partial_i \times I$ with $D^2 \times I$ one must attach $* \times I$ to $p_i \times I$ and fill with the identity at the ends $\partial_i \times 0$ and $\partial_i \times 1$. Now, h has been extended to D^2 and is concretely realized as $D^2 \times I$ by sticking the ends together as when closing a braid; call this intermittent mapping torus $M(h)$ which is diffeomorphic to $D^2 \times S^1$. Now, extending the map to D^3 is trivial (again, $h|_{\partial\Omega_n} = id$) and one immediately sees that the 2-handle attached corresponding to ∂_0 cancels the 1-handle from the open book construction. Moreover, this cancellation occurs without disturbing the rest of the boundary of $W(H)$. Thus, we are left with a 0-handle (i.e. D^4) with boundary S^3 containing a very nice copy of $M(h)$. To obtain $W^4(r)$ we now attach the remaining n 2-handles to D^4 along the copies of $D^2 \times S^1$ in $M(h)$ in the canonical way.

Summarizing the previous two paragraphs, an Artin presentation r determines a framed pure braid β in $\mathbb{R}^3$ (which is the same as in S^3) and $W^4(r)$ is obtained from D^4 by attaching 2-handles according to β. In the language of the Kirby calculus, all $W^4(r)s$ are '2-handlebodies' ([GS], p.124). For more on the manifolds $W^4(r)$ see section 4.

Remark (2.1.1). One subtle but important distinction that must be made here between an $r \in \mathcal{R}_n$ and a framed pure braid in $S^3 = \partial D^4$ is that in an Artin presentation the framings are canonically included (they are not 'put in by hand' as in the Kirby calculus) thus, e.g. avoiding serious self-linking problems [Wi1], p.363. In fact, a moment of reflection by the reader should reveal that without this 'canonicity' one would not obtain the purely group theoretic analogue of Donaldson's theorem [W1], p.240 Theorem 1, and its important consequences. See also [W1], p.241 and [W3].

Hence, one tack to obtain an Artin presentation for a specific 4-manifold is to obtain a surgery diagram for the manifold that is a framed pure braid in S^3 and then determine the corresponding Artin presentation from this framed pure braid. Of course, saying an Artin presentation r gives a *closed* 4-manifold X^4 means that $M^3(r) = S^3$ and $W^4(r) \cup D^4 = X^4$ (i.e. close up with a 4-handle). We pursue this tack in sections 2.2-2.4 below. We abuse notation and say an Artin presentation or a surgery diagram gives a closed 4$-$manifold when it actually presents the closed manifold minus the interior of a 4-handle (which can only be attached in one way, so there is no ambiguity).

We close this section by recalling useful knot theoretic structures in AP Theory. The simplicity of these structures allows us to avoid doing surgery 'by hand', avoids self-linking problems, etc. by use of a computer algebra system such as MAGMA and significantly adds to the power of AP Theory. We point out that, as usual, everything is group theoretic.

Fix $r \in \mathcal{R}_n$, $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$, with $\det A(r) = \pm 1$, in particular $\Sigma^3(r)$ is a $\mathbb{Z}$-homology 3-sphere. There are $n+1$ distinguished knots in $\Sigma^3(r)$ that are defined by the boundary circles $\partial_0, \ldots, \partial_n$ of Ω_n and we denote these knots by $k_0, \ldots, k_n$. Let c_i denote the complement of k_i in $\Sigma^3(r)$ and let G_i denote the fundamental group of c_i. Since $A(r)$ is unimodular, $A(r)^{-1}$ is also a symmetric integer matrix and, in fact, is the linking matrix of the knots k_i,

$i = 1, \ldots, n$. We let b_{ij} denote the ij^{th} entry of $A(r)^{-1}$ (abbreviating b_{ii} to just b_i) and let $s = \sum_{ij} b_{ij}$. In $\Sigma^3(r)$, the self linking number of k_0 is s and of k_i, $i \neq 0$, is b_i. We let m_i, l_i denote the peripheral structure of the knot k_i, which consists of two special commuting elements in G_i, where m_i is a meridian of k_i and l_i is homologically trivial in the complement of k_i. Then, we have:

$$\begin{aligned} G_0 &= \langle x_1, \ldots, x_n \mid r_1 = r_2 = \cdots = r_n \rangle, \\ m_0 &= \text{any } r_i, \\ l_0 &= x_1 x_2 \cdots x_n m_0^{-s}, \end{aligned}$$

and for $i = 1, \ldots, n$ we have:

$$\begin{aligned} G_i &= \langle x_1, \ldots, x_n \mid r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n \rangle, \\ m_i &= r_i, \\ l_i &= x_i m_i^{-b_i}. \end{aligned}$$

Two remarks are in order. First of all, we get *all* knots and links in any arbitrary closed, orientable 3−manifold this way (González-Acuña unpublished). Second, the definition given here of G_i for $i \neq 0$ appears to be slightly different from that given in [W1], p.227, but in fact the two are equivalent (this was pointed out to the second author by González-Acuña). This follows since the Artin Condition (AC) implies that in G_i (definition given here) we have:

$$x_1 x_2 \cdots x_n = x_1 x_2 \cdots x_{i-1} (r_i^{-1} x_i r_i) x_{i+1} \cdots x_n,$$

which immediately implies that $x_i = r_i^{-1} x_i r_i$ in G_i. That is, $(x_i, r_i) = 1$ in G_i (where (a, b) is MAGMA notation for the commutator $a^{-1}b^{-1}ab$), showing the two definitions are equivalent. In fact, for $i \neq 0$, m_i and l_i commuting in G_i is equivalent to x_i and r_i commuting in G_i.

(2.2) Pure braid for $E(n)$. Our starting point is the framed link diagram in [HKK], p.66 (see also [GS], p.305) that presents a 2-handlebody with boundary S^3 and gives $E(n)$ upon closing up with a 4-handle. (As mentioned earlier, we abuse notation and say this diagram presents $E(n)$ where no confusion should arise.) By straightforward isotopy of the outer strand (the trefoil) we obtain Figure 1. The two large bands both represent $6n - 2$ strands, each strand with framing -2. A box containing '-1' represents a twist of all strands (as when twisting ribbon) in the direction corresponding to a negative crossing in our orientation convention in Figure 8. We refer to the trefoil in Figure 1 as T and to the small circle linking it as S, which have framings 0 and $-n$ respectively.

All circles formed by closing a pure braid are individually not knotted, so the first step is to unknot the trefoil T. To accomplish this, one performs a 2-handle slide on T; in practice this corresponds to performing a band sum of T with a parallel curve to another knot K representing the framing on K (see [GS], pp.141-143). Here we slide T over the innermost circle in the left large band using the trivial band as in Figure 2. One checks that the curve in Figure 2 that T is being band summed with is a parallel curve to the innermost strand and has linking number -2 with it (don't forget the '-1' box!). Let T' denote the result of 2-handle sliding T. Figure 3 is obtained from Figure 2 by isotopy, in particular grab the part of T' in Figure 2 that hangs below the two

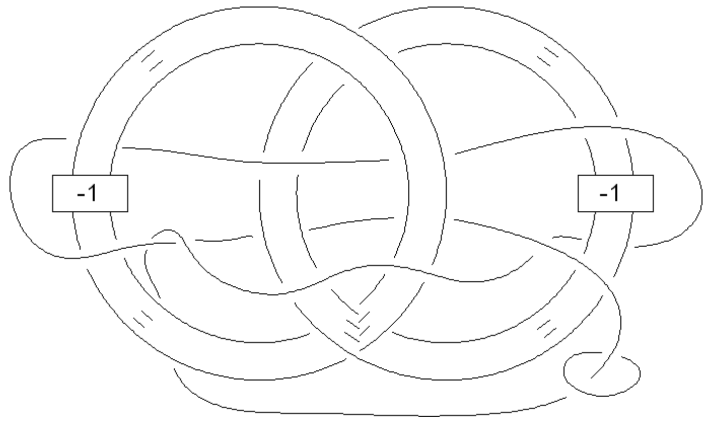

Figure 1. Surgery diagram for $E(n)$. The large bands represent $6n-2$ strands and all framings equal -2, except the trefoil T with framing 0 and the small circle S linking it with framing $-n$.

large bands and swing it back and then up (other minor changes by isotopy here should be obvious). Straightforward isotopy of Figure 3 produces Figure 4 where it is apparent that T' is not knotted.

It does not seem possible to isotop Figure 4 to a pure braid, so we perform another 2-handle slide. This time, slide T' over the outermost strand in the right large band (again using a trivial band to band sum with) as shown in Figure 5. After a little isotopy one obtains Figure 6 (ignoring the hatched rectangle for the moment). Let T'' denote the result in Figure 6 of sliding T' (S is unchanged).

Now, Figure 6 isotops nicely to a pure braid. To see this, take the hatched rectangle in Figure 6, grab its upper left long boundary edge and pull it around, making a rather large (ambient) expansion of the hatched rectangle into a large backwards 'C' shape (the short dimension of the hatched rectangle extends

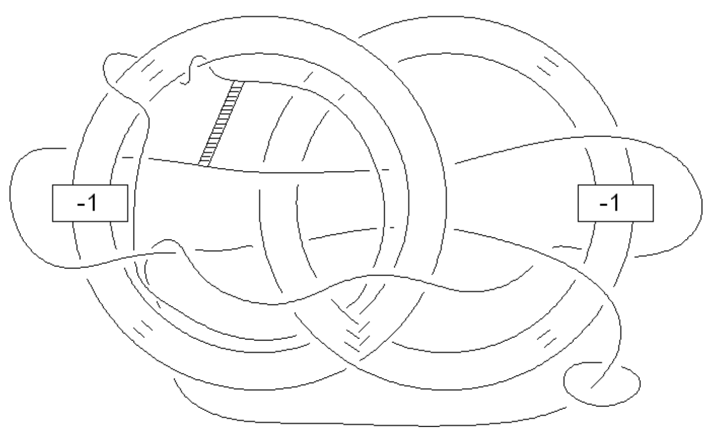

Figure 2. A $2-$handle slide of T over the innermost curve in the left large band using the indicated parallel curve and dashed band.

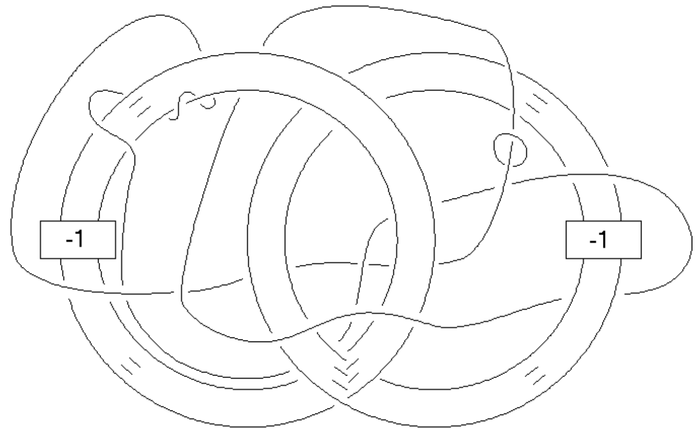

Figure 3. The result T' of 2-handle sliding T.

and bends around). Except for S, one now has a pure braid. A little more straightforward isotopy produces Figure 7, which is a pure braid for $E(n)$. The hatched rectangle does not appear in Figure 7, but one imagines it bending around on the right-hand side to close the braid. Figure 7 contains a total of $12n-2$ strands: the two large bands each represent $6n-2$ strands (each strand therein has framing -2), the $(12n-3)^{\text{rd}}$ strand (second from the right) is T'', and the $(12n-2)^{\text{nd}}$ strand (right-most) is S with framing $-n$.

It remains to determine the framing on T'' (this is the only one that changed), which is calculated using the formula in [GS] p.142. The first 2−handle slide results in T' with framing -2 since the relevant (signed, according to handle addition or subtraction) linking number is 0. The second 2−handle slide results in T'' with framing still -2 since in this case the relevant signed linking number (whose overall sign is independent of orientation choices) is equal to

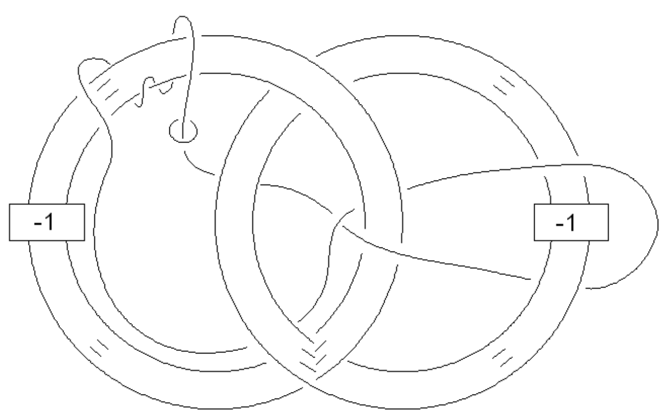

Figure 4. The result of isotoping T' (and S), which is not knotted.

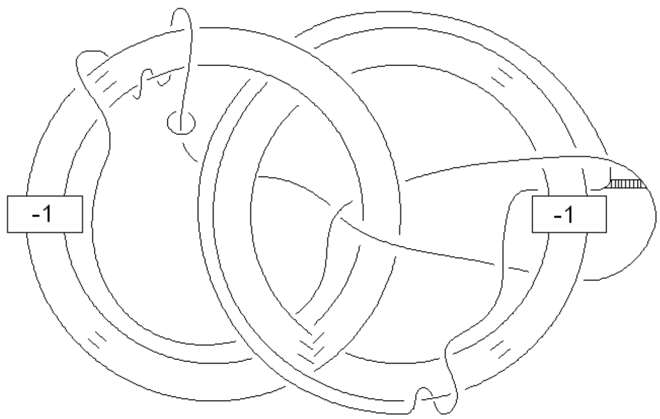

Figure 5. A 2-handle slide of T' over the outermost curve in the right large band using the indicated parallel curve and dashed band.

$+1$ implying $\pm 2lk(\cdot,\cdot) = +2$. Thus, in the pure braid diagram for $E(n)$ in Figure 7 all framings equal -2 except for the right-most strand which has framing $-n$. In particular, for the Kummer surface $E(2)$ all framings equal -2.

Remark (2.2.1). In Figure 1, the two large bands together form the compactified Milnor fiber $M_c(2,3,6n-1)$ with boundary the Seifert fibered $\mathbb{Z}$-homology 3$-$sphere $\Sigma(2,3,6n-1)$ and the trefoil union the small circle linking it form the Gompf nucleus $N(n)$ (see [GS], sec. 3.1, 6.3, 7.3 and 8.3). It is clear from the above that all Milnor fibers $M_c(2,3,6n-1)$ admit Artin presentations.

(2.3) An Algorithm. Given a framed pure braid in $\mathbb{R}^3$, we wish to construct the corresponding Artin presentation. To make this explicit, we must fix some conventions. We will use β to denote both a braid and a framed braid, where no confusion should arise. As usual, braids will be drawn as generic diagrams

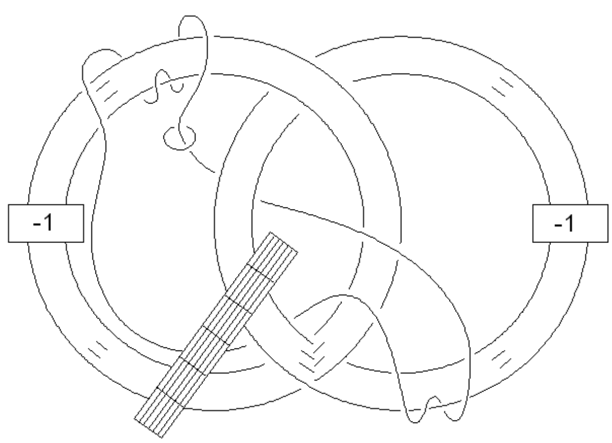

Figure 6. The result T'' of 2-handle sliding T. The hatched rectangle will be used to isotop to a pure braid.

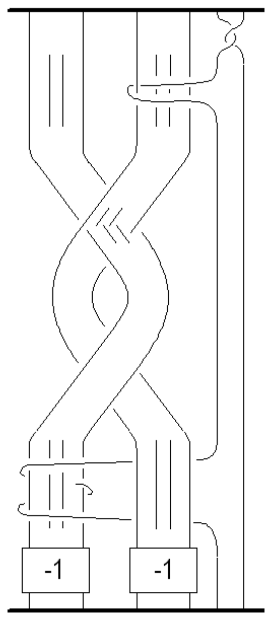

Figure 7. Pure braid for $E(n)$. The large bands represent $6n - 2$ strands and all framings equal -2, except for the rightmost strand with framing $-n$.

in the plane with the strands ordered $1, 2, \ldots, n$ from left to right. We read our braids upwards, especially when composing them. In particular, each strand is oriented up. For a pure braid β, C_i will denote the oriented circle consisting of the i^{th} strand and the trivial segment that would close that strand upon closing the braid (the orientation is inherited from that of the corresponding braid strand). Crossings in any oriented generic link diagram in the plane are assigned a sign as in Figure 8. If C_1 and C_2 are two oriented circles in a generic link diagram in the plane, then their linking number $lk(C_1, C_2)$ is defined to be the number of positive undercrossings of C_2 under C_1 minus the number of negative undercrossings of C_2 under C_1. The linking number is well defined and symmetric (see [GS] sec. 4.5). For an n-strand framed pure braid β the linking matrix $L(\beta)$ of β is the $n \times n$ symmetric integer matrix L where $L_{ij} = lk(C_i, C_j)$ for $i \neq j$ and equals the framing coefficient of C_i for $i = j$. Similarly, one can define the linking matrix of any ordered oriented framed generic link diagram in the plane.

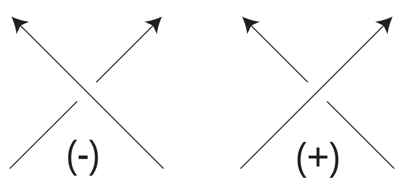

Figure 8. Crossing signs in an oriented link diagram.

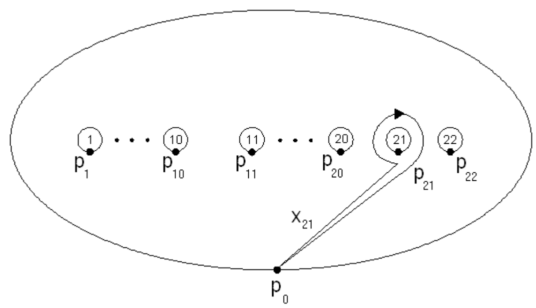

Figure 9. Ω_{22} with basepoints $p_0, \ldots, p_{22}$ on boundary components $\partial_0, \ldots, \partial_{22}$. Also indicated is a generator x_{21} of $\pi_1(\Omega_{22}, p_0)$.

Remark (2.3.1). If $r \in \mathcal{R}_n$ corresponds to β a framed pure braid then $A(r) = L(\beta)$. This follows from [W1], section 1 and [GS], p. 125. We note that orientations/conventions fixed agree with both [W1] and [GS].

Any pure braid $\beta \in P_n$ can be written as a product of Dehn twists about simple closed curves in Ω_n. Thus, we will need these three steps:

Step I. Given a pure braid β resulting from a single Dehn twist, determine the corresponding Artin presentation.

Step II. Compose two Artin presentations.

Step III. Correct Framings.

Remark (2.3.2). Again, Step III is necessary since when going from a framed pure braid (where framings are not canonically included) to an Artin presentation (where framings are canonically included) an ad hoc framing correction must be made at some point.

We describe these in detail.

Step I. First, $\pi_1(\Omega_n, p_0)$ has canonical generators. Figure 9 shows Ω_{22} with basepoint p_0 and the generator x_{21} (the other generators are similar; see also [W1] p. 225 and p. 244). Also depicted in Figure 9 are basepoints on the boundary components $\partial_1, \ldots, \partial_{22}$ (as referred to in section 2.1).

We use two examples to illustrate this step. For the first example, take the Dehn twist depicted in Figure 10 about the oriented simple closed curve D_1 (for

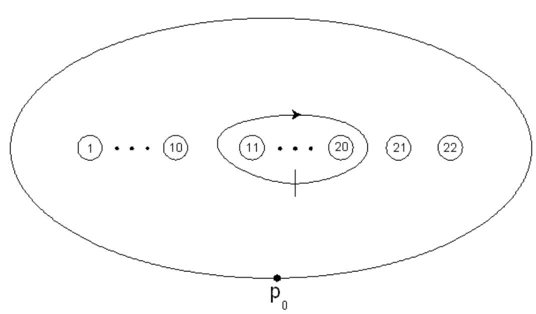

Figure 10. Ω_{22} with an oriented simple closed curve D_1 and a small segment laid across it.

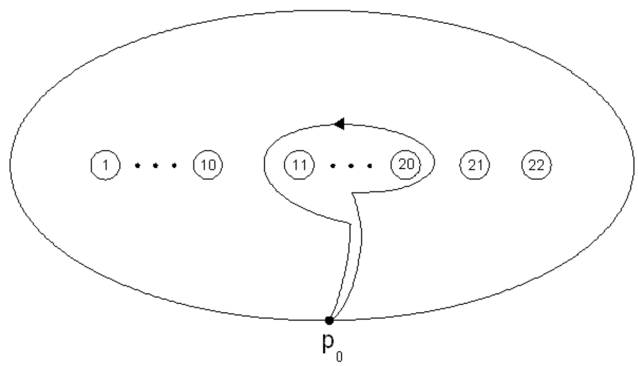

Figure 11. Ω_{22} with a loop representing $r_{11}, \ldots, r_{20}$.

the moment ignore the small segment laid across D_1). Usually one would take a cylinder neighborhood $S^1 \times [-1, 1]$ of D_1 in Ω_{22} and replace it with a twisted version (often a cut along D_1 takes place) according to some fixed orientation convention (see, for example, [GS] p.295). Following the motivation setforth in section 2.1, we prefer to realize the Dehn twist canonically as an isotopy of Ω_{22} in $\mathbb{R}^3$ as follows. Start with a copy of Ω_{22} (as in Figure 10) laying flat on the (possibly imaginary) table in front of you and a small cylinder neighborhood $N = S^1 \times [-1, 1]$ of D_1 in Ω_{22}. The inner boundary curve of N bounds a compact disk with 10 holes denoted Ω'_{10}. Slowly raise Ω_{22} up off the table and while doing so grab Ω'_{10} and slowly rotate it clockwise about a central point (with the cylinder neighborhood N stretching like rubber) one complete revolution. If one pictures the paths traced out by the center points of the 22 punctures in Ω_{22} during this Dehn twist, one immediately sees the pure braid obtained from Figure 7 with $n = 2$ by just taking the '-1' box on strands $11 - 20$ and taking the remaining strands to be trivial. This Dehn twist, realized as an isotopy, gives a self diffeomorphism h of Ω_{22} that is fixed on $\partial\Omega_{22}$, namely the time 1 map of the isotopy. As discussed above in section 2.1 and [W1] pp. 243–244, the automorphism $h_\#$ of $\pi_1(\Omega_{22}, p_0) \cong F_{22}$ induced by h is of the form $x_i \mapsto r_i^{-1} x_i r_i$ for some words r_i and $r = \langle x_1, \ldots, x_{22} \mid r_1, \ldots, r_{22} \rangle$ is our desired Artin presentation. The word r_i is nontrivial ($\neq 1$) only for $i = 11, \ldots, 20$ and these are all equal to one another. To compute r_{11}, say, lay a straight segment across D_1 in front of ∂_{11} as in Figure 10 and follow the segment through the isotopy above. After the isotopy, add two oriented edges to the isotoped segment: one from p_0 to the upper endpoint and one from the lower endpoint to p_0 as in Figure 11; the word in $\pi_1(\Omega_{22}, p_0)$ represented by this oriented loop is $r_{11} = x_{20}^{-1} x_{19}^{-1} \cdots x_{11}^{-1}$.

We note two important points concerning the above example. First, it conveyed the orientation convention of Dehn twists used here, namely grab the inner compact disk with holes and twist it in the direction of the arrow on the curve one is twisting about. Second, the small segment laid across D_1 formed the 'meat' of the relations and only crossed D_1 once. When computing r_i in general, one must choose this segment to traverse all occurrences of the curve

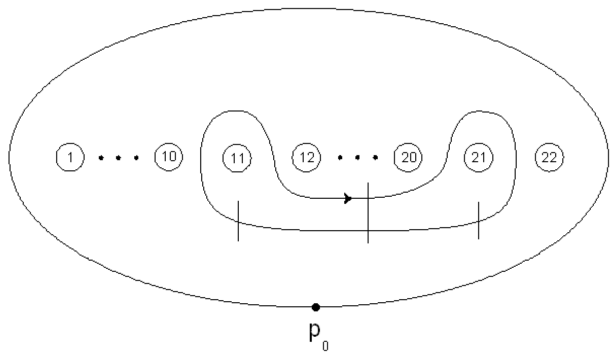

Figure 12. Ω_{22} with an oriented simple closed curve D_{24} and three small segments laid across it.

one is twisting about between a nice path (usually a straight line segment or a small isotopy of one) from p_0 to p_i. This is shown in the following example.

For this example, take the Dehn twist depicted in Figure 12. The automorphism of F_{22} is clearly the identity on $x_1, \dots, x_{10}, x_{22}$. Figure 13 shows the loop representing both words $r_{11} = r_{21} = x_{21}^{-1} x_{11}^{-1}$ (as the reader can verify using the two small segments in Figure 12 that cross D_{24} once). The more interesting relations are $r_{12}, \dots, r_{20}$, which are all equal to one another. To compute these one must use a segment that crosses D_{24} twice, such as the middle segment in Figure 12. The resulting loop is shown in Figure 14 and represents the word $x_{21} x_{11} x_{21}^{-1} x_{11}^{-1}$. This completes Step I.

Step II. Our data is two Artin presentations r, r' arising from Dehn twists about D, D' with corresponding h, h' and $h_\#$, $h'_\#$. Then, the composite Artin presentation $r'' = r' \circ r$ is obtained using the formula (see [W1], p.245):

$$r''_i = r'_i \cdot h'_\#(r_i).$$

Step II is impractical by hand when the presentations are not small and use of a computer algebra system, such as MAGMA, is invaluable.

Step III. Our data now is a framed pure braid β and an Artin presentation r' resulting from repeated applications of Steps I and II. One also has the

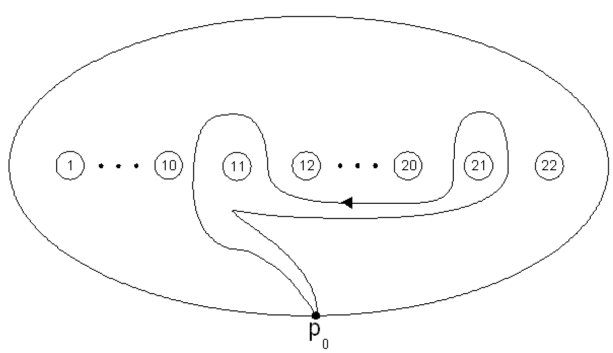

Figure 13. Ω_{22} with a loop representing r_{11} and r_{21}.

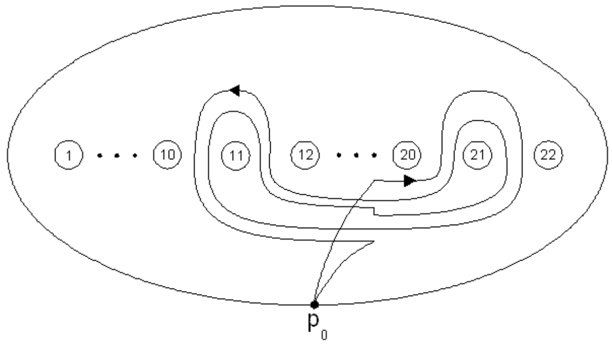

Figure 14. Ω_{22} with a loop representing $r_{12}, \ldots, r_{20}$.

matrices $L(\beta)$ and $A\left(r'\right)$ which differ only possibly on their diagonals. One corrects (see Remark (2.3.2) and Section 2.1) using the simple rule:

$$\text{let } \delta_i = L(\beta)_{ii} - A(r')_{ii}, \text{ and}$$
$$\text{let } r_i = x_i^{\delta_i} \cdot r_i'.$$

The result is the Artin presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ and $A(r) = L(\beta)$. We point out that when correcting framings one must multiply on the left by the corresponding $x_i^{\delta_i}$, otherwise the resulting presentation is usually not Artin. This completes Step III.

(2.4) Artin presentation of $K3$. Begin with the framed pure braid in Figure 7 with $n = 2$. Call this braid β and recall that all framings equal -2. We need a series of Dehn twists producing β (ignoring framings for the moment). To take care of β (reading up from the bottom) up until the point where the two large bands first cross each other, perform Dehn twists about D_1, D_2, D_3, and D_4 (in that order!) as in Figure 10 and Figures 15–17. (It may seem that the '-1' on the left band has been left off, but the reader should check that this is not the case.) Now we attack the brunt of β consisting of the 'Milnor fiber' where the two large bands cross each other and then intertwine. For this part we will need

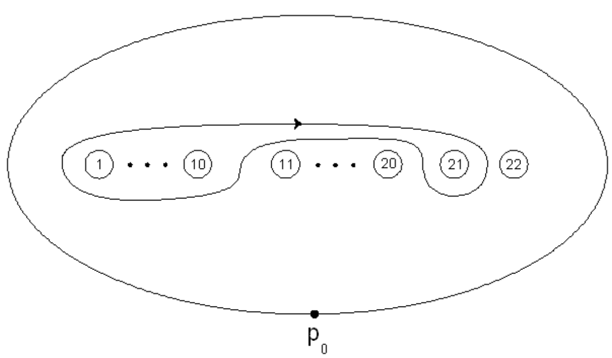

Figure 15. Ω_{22} with an oriented simple closed curve D_2.

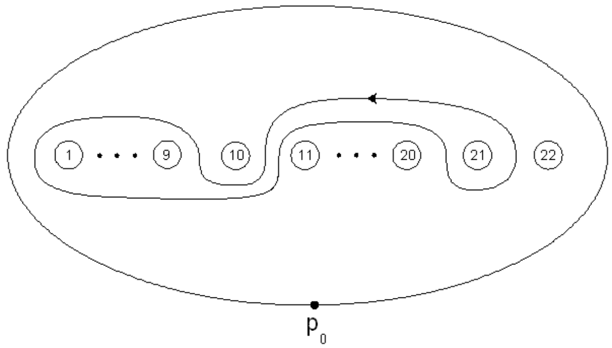

Figure 16. Ω_{22} with an oriented simple closed curve D_3.

Figures 18 and 19 repeated in an alternating fashion. Figure 18 represents D_5, D_7, $D_9, \ldots, D_{23}$ where D_{5+2j}, $j = 0, 1, 2, \ldots, 9,$ corresponds to Figure 18 with $k = j + 1$ and $k' = j + 11$. Figure 19 represents $D_6, D_8, D_{10}, \ldots, D_{22}$ where D_{6+2j}, $j = 0, 1, 2, \ldots, 8$, corresponds to Figure 19 with $k = j + 1$. Then, one performs Dehn twists about the following ordered and oriented curves: $D_5, D_6, \ldots, D_{22}, D_{23}$. The reader should check that this series of Dehn twists performs as claimed. To finish up, one twists about D_{24} as in Figure 12 and then about D_{25} as in Figure 20. This series of Dehn twists gives β up to framings.

Now, using Step I from section 2.3, one writes down the Artin presentation corresponding to each of the Dehn twists in this series. We organize this data into a 25×22 array R of words in F_{22} where $R[i, \cdot]$ corresponds to D_i (i.e. $R[i, j]$ is the j^{th} relation of the i^{th} Artin presentation). Assume that R is initialized as the 25×22 array of identity elements in F_{22}. The nontrivial elements in R are as follows.

	$R[1, i]$
$i = 11, \ldots, 20$	$x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$

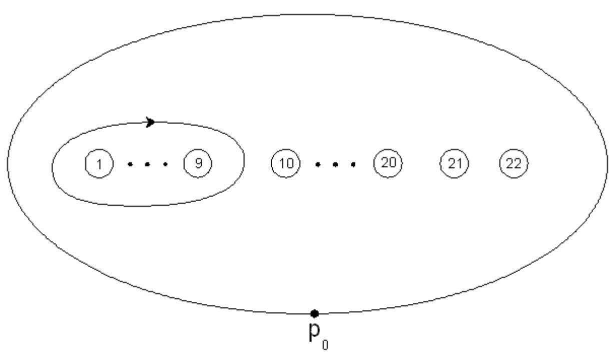

Figure 17. Ω_{22} with an oriented simple closed curve D_4.

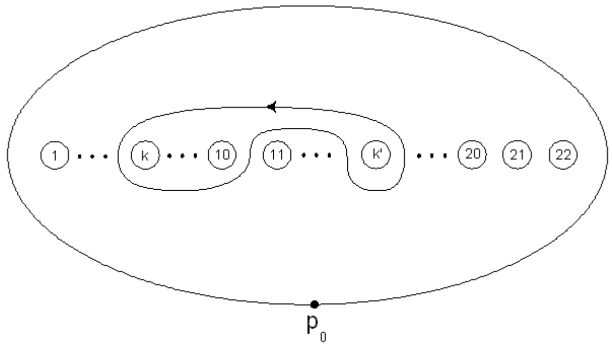

Figure 18. Ω_{22} with an oriented simple closed curve D_*.

$R[2,i]$	
$i=1,\ldots,10$	$x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}x_{19}x_{20}$ $x_{21}^{-1}x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$ $x_{10}^{-1}x_9^{-1}x_8^{-1}x_7^{-1}x_6^{-1}x_5^{-1}x_4^{-1}x_3^{-1}x_2^{-1}x_1^{-1}$
$i=21$	$x_{21}^{-1}x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$ $x_{10}^{-1}x_9^{-1}x_8^{-1}x_7^{-1}x_6^{-1}x_5^{-1}x_4^{-1}x_3^{-1}x_2^{-1}x_1^{-1}$ $x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}x_{19}x_{20}$

$R[3,i]$	
$i=1,\ldots,9$	$x_1x_2x_3x_4x_5x_6x_7x_8x_9$ $x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}x_{19}x_{20}x_{21}$ $x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$
$i=10$	$x_9^{-1}x_8^{-1}x_7^{-1}x_6^{-1}x_5^{-1}x_4^{-1}x_3^{-1}x_2^{-1}x_1^{-1}$ $x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}x_{19}x_{20}$ $x_{21}^{-1}x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$ $x_1x_2x_3x_4x_5x_6x_7x_8x_9$ $x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}x_{19}x_{20}x_{21}$ $x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$
$i=21$	$x_{20}^{-1}x_{19}^{-1}x_{18}^{-1}x_{17}^{-1}x_{16}^{-1}x_{15}^{-1}x_{14}^{-1}x_{13}^{-1}x_{12}^{-1}x_{11}^{-1}$ $x_1x_2x_3x_4x_5x_6x_7x_8x_9$ $x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}x_{19}x_{20}x_{21}$

$R[4,i]$	
$i=1,\ldots,9$	$x_9^{-1}x_8^{-1}x_7^{-1}x_6^{-1}x_5^{-1}x_4^{-1}x_3^{-1}x_2^{-1}x_1^{-1}$

Now, the relations $R[5-23,i]$ lend themselves well to looping/shorthand (which we utilize especially when using MAGMA). Let $w = x_{19}^{-1}x_{18}^{-1}\cdots x_{11}^{-1}$ and let w_j denote the first j letters of w read from the right for $j=0,\ldots,9$. For example, $w_0 = 1$ (i.e. the identity in F_n) and $w_2 = x_{12}^{-1}x_{11}^{-1}$. Then, $R[5,i]$, $R[7,i]$, $R[23,i]$ are defined by the following where $j=0,1,\ldots,9$:

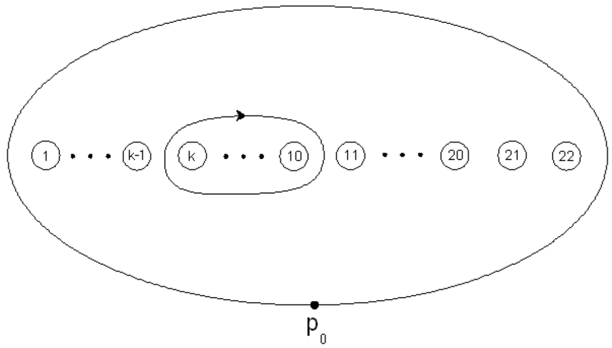

Figure 19. Ω_{22} with an oriented simple closed curve D_*.

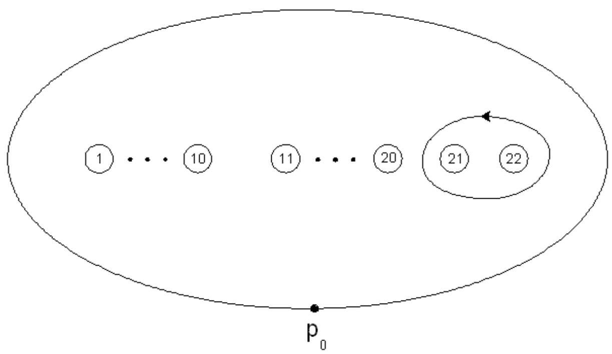

Figure 20. Ω_{22} with an oriented simple closed curve D_{25}.

$R[5+2j,i]$	
$i=(j+1),\ldots,10$	$x_{j+1}x_{j+2}\cdots x_{11+j}w_j$
$i=11+j$	$w_jx_{j+1}x_{j+2}\cdots x_{11+j}$

Also, $R[6,i]$, $R[8,i]$, $R[22,i]$ are defined by the following where $j=0,1,\ldots,8$:

$R[6+2j,i]$	
$i=(j+1),\ldots,10$	$x_{10}^{-1}x_9^{-1}\cdots x_{j+1}^{-1}$

And the last two Artin presentations:

$R[24,i]$	
$i=11,21$	$x_{21}^{-1}x_{11}^{-1}$
$i=12,\ldots,20$	$x_{21}x_{11}x_{21}^{-1}x_{11}^{-1}$

$R[25,i]$	
$i=21,22$	$x_{21}x_{22}$

The list of Artin presentations corresponding to the series of Dehn twists given above is complete. Now, one simply composes these 25 presentations (with MAGMA!) using a loop statement and the formula from Step II in section 2.3. Call the result of this iterated composition r'. To correct the framings, one computes the exponent sum matrix of r' (again using MAGMA) and checks the diagonal of this matrix which is (starting from the upper left):

$$(\underbrace{-1,\ldots,-1}_{9\text{ times}},0,-1,\underbrace{0,\ldots,0}_{10\text{ times}},1)$$

To make these entries all equal -2, one corrects r' using Step III calling the result r. This is the desired Artin presentation for the Kummer surface $K3$.

After obtaining r with MAGMA, one immediately checks that the presentation is in fact Artin. To do so, simply prompt MAGMA to compute the right hand side of the Artin condition (AC). The result should be (and for our r is) the left hand side of (AC). This is an important test, but it is also a test that MAGMA can always carry out as the word problem in F_n is solved and MAGMA must only freely reduce.

By construction, $M^3(r)$ is S^3 and $W^4(r)$ is $K3$. Despite the length of the presentation r (which is given below) MAGMA readily verifies that $\pi(r) = 1$. To look at $W^4(r)$ one proceeds to $A(r)$ which appears in Figure 21. This matrix is even, unimodular, has 19 negative eigenvalues and 3 positive ones, hence is $\mathbb{Z}$-congruent to $2E_8 \oplus 3H$ as expected. One is now ready to reap the rewards of this work. The Artin presentation r can be easily and orderly investigated with MAGMA where nothing has to be done by hand and one doesn't need to worry about surgery diagrams, etc. Examples appear in the following section.

The inverse matrix of $A(r)$, which appears in Figure 22, provides the peripheral structure of the knots k_i, $i = 0,\ldots,22$, described at the beginning of this section. Notice that the diagonal consists entirely of -2, 0, and 2, which as a consequence immediately again gives Artin presentations for the appropriate $(1,\pm1)$ Dehn spheres. Further, notice that the total sum of $A(r)^{-1}$, denoted s, equals -6, another computational advantage.

The knots k_i are nontrivial only for $i = 0, 10, 11, 21, 22$; k_{10} and k_{11} are 5_2s, k_{22} is a trefoil, and k_{21}, with Alexander polynomial $\Delta = t^4 - t^2 + 1$, is a cable of the trefoil. However, k_0 has Alexander polynomial $\Delta = t^8 - 2t^7 - 5t^5 + 13t^4 - \ldots$ and is off the usual knot tables; its 2, 3, 4, 5 torsion is given by (29), $(13, 13)$, $(15, 435)$, $(251, 251)$.

It seems curious that here the only non-fibered knots are k_{10} and k_{11}, precisely where the pair of $3s$ appears off the diagonal in $A(r)^{-1}$ (Figure 22); see also the end of section 2.1.

As $\mathcal{R}_{22}$ is a group, one may wish to compute r^{-1}. To do so, one performs the same series of Dehn twists as for r but in the reverse order and with reverse orientation. One must repeat Step I for all of these Dehn twists and the work is equivalent to the work involved with getting r. After doing so, one compares the lengths of the relations in r and r^{-1} which appear below. (We use $\#r$ to denote the total length of all relations.) We note that shorter presentations are not necessarily more useful computationally, especially with MAGMA, as one quickly finds.

–2	–1	–1	–1	–1	–1	–1	–1	–1	–1	1											
–1	–2	–1	–1	–1	–1	–1	–1	–1	–1	1	1										
–1	–1	–2	–1	–1	–1	–1	–1	–1	–1	1	1	1									
–1	–1	–1	–2	–1	–1	–1	–1	–1	–1	1	1	1	1								
–1	–1	–1	–1	–2	–1	–1	–1	–1	–1	1	1	1	1	1							
–1	–1	–1	–1	–1	–2	–1	–1	–1	–1	1	1	1	1	1	1						
–1	–1	–1	–1	–1	–1	–2	–1	–1	–1	1	1	1	1	1	1	1					
–1	–1	–1	–1	–1	–1	–1	–2	–1	–1	1	1	1	1	1	1	1	1				
–1	–1	–1	–1	–1	–1	–1	–1	–2	–1	1	1	1	1	1	1	1	1	1			
–1	–1	–1	–1	–1	–1	–1	–1	–1	–2	1	1	1	1	1	1	1	1	1	1	–1	
1	1	1	1	1	1	1	1	1	1	–2	–1	–1	–1	–1	–1	–1	–1	–1	–1	–1	
	1	1	1	1	1	1	1	1	1	–1	–2	–1	–1	–1	–1	–1	–1	–1	–1		
		1	1	1	1	1	1	1	1	–1	–1	–2	–1	–1	–1	–1	–1	–1	–1		
			1	1	1	1	1	1	1	–1	–1	–1	–2	–1	–1	–1	–1	–1	–1		
				1	1	1	1	1	1	–1	–1	–1	–1	–2	–1	–1	–1	–1	–1		
					1	1	1	1	1	–1	–1	–1	–1	–1	–2	–1	–1	–1	–1		
						1	1	1	1	–1	–1	–1	–1	–1	–1	–2	–1	–1	–1		
							1	1	1	–1	–1	–1	–1	–1	–1	–1	–2	–1	–1		
								1	1	–1	–1	–1	–1	–1	–1	–1	–1	–2	–1		
									1	–1	–1	–1	–1	–1	–1	–1	–1	–1	–2		
									–1	–1										–2	1
																				1	–2

Figure 21. $A(r)$ for r representing the Kummer surface.

Relation	r	r^{-1}	Relation	r	r^{-1}
1	130	176	12	252	502
2	131	403	13	247	501
3	132	628	14	240	500
4	133	851	15	231	499
5	134	1072	16	220	498
6	135	1291	17	207	497
7	136	1508	18	192	496
8	137	1723	19	175	495
9	138	1936	20	156	494
10	644	2126	21	529	573
11	258	108	22	5	383

Total Relator Length	
$\#r$	$\#r^{-1}$
4562	17260

In the following, we denote the just constructed r, r^{-1} by $k3, k3^{-1}$. Let t_1 be the Torelli of [W1] p. 228. If we multiply $k3^{-1}$ "at 20 by t_1" [W1] p.227,

–2		1	1		–1	–1		1	1	–1	1	1		–1	–1		1	1			
	–2		1	1		–1	–1		1	–1	–1	1	1		–1	–1		1	1		
1		–2		1	1		–1	–1			–1	–1	1	1		–1	–1		1		
1	1		–2		1	1		–1	–1	1		–1	–1	1	1		–1	–1			
	1	1		–2		1	1		–1	1	1		–1	–1	1	1		–1	–1		
–1		1	1		–2		1	1			1	1		–1	–1	1	1		–1		
–1	–1		1	1		–2		1	1	–1		1	1		–1	–1	1	1			
	–1	–1		1	1		–2		1	–1	–1		1	1		–1	–1	1	1		
1		–1	–1		1	1		–2			–1	–1		1	1		–1	–1	1		
1	1		–1	–1		1	1			3		–1	–1		1	1		–1	–1	–2	–1
–1	–1		1	1		–1	–1		3			1	1		–1	–1		1	1	–2	–1
1	–1	–1		1	1		–1	–1			–2		1	1		–1	–1		1		
1	1	–1	–1		1	1		–1	–1	1		–2		1	1		–1	–1			
	1	1	–1	–1		1	1		–1	1	1		–2		1	1		–1	–1		
–1		1	1	–1	–1		1	1			1	1		–2		1	1		–1		
–1	–1		1	1	–1	–1		1	1	–1		1	1		–2		1	1			
	–1	–1		1	1	–1	–1		1	–1	–1		1	1		–2		1	1		
1		–1	–1		1	1	–1	–1			–1	–1		1	1		–2		1		
1	1		–1	–1		1	1	–1	–1	1		–1	–1		1	1		–2			
	1	1		–1	–1		1	1	–1	1	1		–1	–1		1	1		–2		
									–2	–2										2	1
									–1	–1										1	

Figure 22. The inverse matrix $A(r)^{-1}$ providing the peripheral structures of the knots $k_0, \ldots, k_{22}$.

i.e. if we take the Artin presentation $r \in \mathcal{R}_{22}$ where r_i equals 1 for $i < 20$ and equals t_1 written in the variables x_{20}, x_{21}, x_{22} for $i = 20, 21, 22$ and multiply it by $k3^{-1}$, we obtain an Artin presentation, which we denote by $k3^{-1}t1.20$, then π remains trivial and all knot groups stay the same except G_0 whose Alexander polynomial changes from $\Delta = t^8 - 2t^7 - 5t^5 + 13t^4 - \ldots$ to $\Delta = t^{10} - 8t^9 + 14t^8 - 2t^7 - 13t^6 + 15t^5 - \ldots$ (both polynomials are irreducible and the new 2, 3, 4, 5 torsions are given by (9), (65, 65), (3, 3, 9), (899, 899)). Assuming the latter homotopy 3-sphere is actually S^3, we have two a priori different smooth structures on the same underlying topological 4-manifold. (Recall that the Torelli preserve $A(r)$ and Freedman's theorem holds if the boundaries are the same).

Do these smooth structures differ due to, say, the arguments of Fintushel-Stern [FS]*?*

To obtain another Artin presentation for the $K3$ surface, which we denote by $\overline{k3}$ and with inverse $\overline{k3}^{-1}$, we take the pure braid in Figure 7 with $n = 2$ and modify it by an isotopy (the same modification applies to $E(n)$ in general). Take the portion of C_{21} that crosses under the right large band and intertwines with the left large band and simply slide it down to the bottom of the braid and then, using the (not drawn) trivial segments that close the braid, slide it around to the top of the braid. The result is shown in Figure 23. Of course, the framings for this braid are the same as before. Following Steps I-III above we obtain $\overline{k3}$. The isotopy of the braid preserved the order of the strands and hence the

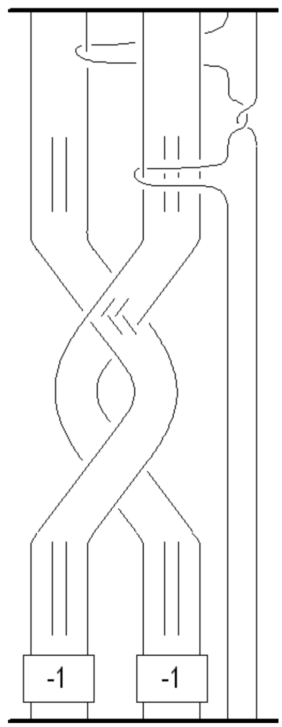

Figure 23. Modified pure braid for $E(n)$.

matrix $A(r)$ for this new presentation is exactly the same as before (Figure 21). For these Artin presentations we have $\#\overline{k3} = 6994$ and $\#\overline{k3}^{-1} = 4398$. We note that $\overline{k3}^{-1}$ is the shortest of the four Artin presentations given here for the Kummer surface.

3. Examples

Thanks to the computer friendly, simple presentations of knot groups and their peripheral structures in AP theory, examples therein need not be laboriously constructed: *they just need to be systematically discovered with MAGMA.* Due to the 'conical', universal structure of AP theory, at least in principle this can at least be done in a systematic, orderly, complete way. Thus, from the beginning AP theory, due to the fact, e.g. that framings need not be put in by hand, automatically and easily yields many of the known interesting examples of classical 3-manifold and knot theory: old and new. From the simplest definition of Poincaré's homology $3-$sphere to examples pertaining to the Cabling conjecture [GAS]. Specifically, at the very beginning [W1] AP theory easily yields cosmetic surgery examples, Luft-Sjerve spheres with fixed point free involutions, failure of Property R in general for $\mathbb{Z}$-homology $3-$spheres, in particular giving boundaries of Mazur manifolds, and nontrivial knots in homotopy $3-$spheres with trivial Alexander polynomial, a phenomenon first discovered by Seifert in the early 1930s.

Using the just constructed Artin presentations of the $K3$ surface, we continue illustrating this natural, canonical flow of instructive examples.

If G is a group, by $ab(G, n)$ we denote the abelianizations of the subgroups of index $\leq n$ (up to conjugation) and we use MAGMA notation, e.g., $ab(G, 4) =$ 1[0], 2[7, 0], 4[2, 2, 0], 4[0, 0], means that G abelianizes to $\mathbb{Z}$ and has, up to conjugation, one subgroup of index 2 which abelianizes to $\mathbb{Z}_7 \times \mathbb{Z}$, no subgroups of index 3, and two subgroups of index 4 abelianizing to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$, respectively.

By, say, $k3^{-1}st24$, we denote the Artin presentation in $\mathcal{R}_{24}$ obtained by not changing r_i of $k3^{-1}$ for $i \leq 22$ and setting $r_{23} = x_{23}$ and $r_{24} = x_{24}$. It is clear (see end of previous section) what, say, $k3^{-1}st24t_3.22 \in \mathcal{R}_{24}$ should be. By $x_i^m r$ we denote the Artin presentation where r_i is changed to $x_i^m r_i$. The Torellis $t_1, t_2, t_3 \in \mathcal{R}_3$ and $T_4' \in \mathcal{R}_4$ are as in [W1] pp.228,229,231. Furthermore, Δ_i denotes the Alexander polynomial of k_i.

I. *Regarding the Cabling Conjecture* [GAS] *in general.* Consider $\Sigma^3(r)$ where $r = x_{22}^{-1}k3^{-1}st24t_3.22 \in \mathcal{R}_{24}$ ($\#r = 17301$); $\pi(r)$ has a balanced (non-Artin) presentation with just three generators:

$$\Big\langle a, b, c \mid c^2 = bcb,\ \ (cbc)^{-1} ab^6 \,(cbc)^{-1} a^{-1}b^{-1}cbc = b^{-6}ab^6 = b^{-2}\,(cbc)^{-1} a^{-1}cbcb^{-6}a\,(ba)^2\, cbc \Big\rangle,$$

and is therefore π-prime in the sense of [GAS], however, the $(1, -1)$ Dehn sphere of the knot k_{21} has fundamental group isomorphic to $I(120) * \pi_1(\Sigma(2, 3, 11))$.

Question: is this Dehn sphere homeomorphic to $\Sigma(2, 3, 5)\#\Sigma(2, 3, 11)$?

The knot k_{21} has the same Alexander polynomial as that of the granny knot in S^3, but their knot groups differ since they have different $ab(\ , 5)$s.

The $(1, 1)$ Dehn sphere of the knot k_3, where $\Delta_3 = t^2 - t + 1$, is simply connected and so $\Sigma^3(r)$ is a $(1, \pm 1)$ Dehn sphere of a knot k in a homotopy 3$-$sphere with Alexander polynomial $\Delta = t^2 - t + 1$, but whose group G has a different $ab(\ , 3)$ than that of the trefoil and is presented by:

$$G = \Big\langle a, b, c \mid bcb = cb^2c,\ b\left(a, \left(b^{-1}a\right)\hat{}\left(b^2\,(bc)^{-1}\,c\,\left(cb\right)^{-1}\right)\right)\Big\rangle.$$

Here, recall that in MAGMA notation $(x, y) = x^{-1}y^{-1}xy$ and $x\hat{}y = y^{-1}xy$. The homology sphere $\Sigma^3(r)$ is the quotient space of a free regular action of $I(120)$ on an M^3 with $H_1(M^3, \mathbb{Z}) = \mathbb{Z}_3^{12}$ and $ab(\pi(r), 15) = ab(I(120), 15)$, however their $ab(\ , 20)$s differ. The Casson invariant, $\lambda\left(\Sigma^3(r)\right)$, of $\Sigma^3(r)$ is ± 1.

Question: is G a knot group of S^3?

II. Tinkering with our Artin presentations for $K3$ seems to give an abundance of $\mathbb{Z}-$homology 3$-$spheres with nontrivial knots where Property R fails, i.e. $G/\langle l\rangle = \mathbb{Z}$ where l is the longitude.

i) k_{10},k_{11} of $\Sigma^3(r)$ where $r = x_1^{-1}x_{22}^{-1}k3^{-1}t_2.1 \in \mathcal{R}_{22}$ ($\#r = 17916$).

ii) k_{20}, k_{22} of $\Sigma^3(r)$ where $r = x_{18}^{-1}k3^{-1}T_4'.19 \in \mathcal{R}_{22}$ ($\#r = 37009$).

iii) k_{15}, k_{22} of $\Sigma^3(r)$ where $r = x_{20}^{-1}k3^{-1}t_3.20 \in \mathcal{R}_{22}$ ($\#r = 44913$).

iv) k_{10}, k_{11} of $\Sigma^3(r)$ where $r = x_{18}^{-1}k3^{-1}t_1.9 \in \mathcal{R}_{22}$ ($\#r = 48643$). Here, $ab(G_{10}, 5) = 1[0], \ldots, 5[0], 5[0, 0], 5[0, 0, 0], 5[2, 0, 0], 5[28371, 0]$. The fundamental group of its $(1, 1)$ Dehn sphere has one single subgroup of index 5 and it abelianizes to $\mathbb{Z}_{28371}$. Such large finite numbers have not appeared before in computations in AP theory. What does their appearance mean?

v) The simplest example seems to be k_{22} of $\Sigma^3(r)$ where $r = x_{20}^{-1}k3^{-1}st23t_3.21 \in \mathcal{R}_{23}$ ($\#r = 27628$). Here $\pi(r)$ and G_{22} are presented by:

$$\begin{aligned} \pi(r) &= \left\langle a, b \mid (aba)^3 = (bab)^2, (ba)^3 = \left(a^{-1}bab\right)^2 \right\rangle, \\ G_{22} &= \left\langle a, b \mid (aba)^3 = (ba)^2 (bab)^{-1} (ab)^2 \right\rangle. \end{aligned}$$

As is well known, the falsity of Property R, i.e. $G/\langle l \rangle = \mathbb{Z}$, implies that the Alexander polynomial is trivial; we also obtain an abundance of nontrivial knots with trivial Alexander polynomials in homotopy 3-spheres (such examples were first discovered by Seifert in the early 1930s): let

$$r = x_{20}^{-1}k3^{-1}st24t_3.22 \in \mathcal{R}_{24}(\#r = 17301),$$

then $\Sigma^3(r)$ is simply connected and $\Delta_{20} \equiv 1$ but $ab(G_{20}, 5) = 1[0], \ldots, 5[0]$, and $5[3, 15, 0]$ repeated 5 times; let $r = x_2\overline{k3}t_2.20 \in \mathcal{R}_{22}(\#r = 11101)$, then $\Sigma^3(r)$ is simply connected and $\Delta_1 \equiv \Delta_{12} \equiv 1$ but $ab(G_1, 5) = ab(G_{12}, 5) = 1[0], \ldots, 5[0]$, $5[0, 0, 0]$, $5[3, 3, 0]$. Here G_{12} is presented by:

$$\left\langle a, b, c \mid \left(a^{-1}, c\right)(c, b)(a, b)\, c = b = \left(c^{-1}, a^{-1}\right)\left(b, c^{-1}\right)(a, b)\left(c^{-1}, a^{-1}\right) \right\rangle.$$

Question: is G_{12} a knot group of S^3?

III. If $r = k3^{-1}t_3.20 \in \mathcal{R}_{22}$ ($\#r = 44550$), then $\Delta_1 \equiv 1$ and $\Delta_2 \equiv 1$ but G_1 and G_2 are not isomorphic since their $ab(\ ,5)$s differ. However, both of their $(1, 1)$ Dehn spheres are simply connected. This illustrates in a different way the phenomenon that 'far away' knots in homotopy 3−spheres can have homeomorphic $(1, 1)$ Dehn spheres [Br].

Unlike with the Donaldson matrices E_8, φ_{4n}, etc., with $K3$ we obtain a much larger amount of knots with $\Delta \equiv 1$. *Is this related to the 'softness' of $K3$ as a Calabi-Yau manifold?*

4. The manifolds $W^4(r)$

We have answered in the affirmative whether all elliptic surfaces $E(n)$ appear as $W^4(r)$s. An open problem is whether every smooth, compact, connected, simply-connected 4-manifold X^4 with a connected, simply-connected boundary $\partial X^4 = M^3$ is a $W^4(r)$. (See [GS] p.344 for a related problem).

In dimension 3, AP theory obtains *all* closed, orientable, connected 3-manifolds and there seem to be no great conceptual difficulties on the horizon in obtaining all Seiberg-Witten invariants of 3−manifolds in AP theory [L], [T] pp.viii,115. Unlike in the simplicial combinatorial case, in AP theory the *same* purely group-theoretic data that determines the 3−manifold, namely r, also *canonically* and *holographically* determines the 4−manifold. Hence, developing 3−dimensional Seiberg-Witten theory in this, its correct, ultimate arena, holds greater promise in further developing also the outstanding open 4−dimensional theory in AP theory.

Similar arguments hold for studying the smoothings of a 4−manifold, à la Fintushel-Stern [FS], using the action of the Torelli, thus generalizing their important work. We remark that, if the 3D Poincaré conjecture were true, then by Freedman's theorem the relation between the Torelli action and smoothings would become even more direct, purely group-theoretic and pristine, perhaps too much so.

Relevant to all of the above is that although finitely presented group theory is considered a difficult subject, the undeniable metamathematical similarities of AP theory with braid theory, holographic dessins ďenfant theory, as well as numerous genuine analogies with Modern Physics, give hope for a definitive, realistic, computer approachable, holographic, and universal approach to X^4 theory [D] p.69, [W2], [W3].

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