

## CONNECTED SUM AT INFINITY AND CANTRELL-STALLINGS HYPERPLANE UNKNOTTING

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Dedicated to Ljudmila V. Keldysh and the members of her topology seminar on the occasion of the centenary of her birth [13].

**1. Introduction.** We give a general treatment of the somewhat unfamiliar operation on manifolds called *connected sum at infinity*, or CSI for short. A driving ambition has been to make the geometry behind the well-definition and basic properties of CSI as clear and elementary as possible. CSI then yields a very natural and elementary proof of a remarkable theorem of Cantrell and Stallings [9, 60]. It asserts unknotting of CAT embeddings of  $\mathbf{R}^{m-1}$  in  $\mathbf{R}^m$  with  $m \neq 3$ , for all three classical manifold categories: topological (TOP), piecewise linear (PL), and differentiable (DIFF)—as defined for example in [36]. It is one of the few major theorems whose statement and proof can be the same for all three categories. We give it the acronym HLT, which is short for “Hyperplane Linearization theorem” (see Theorem 6.1 plus 7.3).

We pause to set out some common conventions that are explained in [36] and in many textbooks. By default, spaces will be assumed to be *metrizable* and *separable* (i.e., having a countable basis of open sets). Simplicial complexes will be unordered. A PL *space* (often called a *polyhedron*) has a maximal family of PL compatible triangulations by locally finite simplicial complexes. A map is *proper* provided the inverse image of each compact set is compact. CAT *submanifolds* will be assumed *properly embedded* and CAT *locally flat*.

This Cantrell-Stallings unknotting theorem (HLT) arose as an enhancement of the more famous Schoenflies theorem initiated by Mazur [39] and completed by Brown [3, 4]. The latter asserts TOP unknotting

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of TOP codimension 1 spheres in all dimensions: any locally flatly embedded  $(m - 1)$ -sphere in the  $m$ -sphere is the common frontier of a pair of embedded  $m$ -balls whose union is  $S^m$ . This statement is cleaner inasmuch as dimension 3 is not exceptional. On the other hand, its proof is less satisfactory, since it does not apply to the parallel PL and DIFF statements. Indeed, for PL and DIFF, one requires a vast medley of techniques to prove the parallel statement, leaving quite undecided the case  $m = 4$ , even today.

The proof of this TOP Schoenflies theorem immediately commanded the widest possible attention and opened the classical period of intense study of TOP manifolds. There is an extant radio broadcast interview of Thom in which he states that, in receiving his Fields Medal in 1958 in Edinburgh for his cobordism theories [63] 1954, he felt that they were already being outshone by Milnor's exotic spheres [42] 1956 and the Schoenflies theorem breakthrough of Mazur just then occurring.

At the level of proofs, the Cantrell-Stallings theorem is perhaps the more satisfactory. The TOP proof we present is equally self contained and applies (with some simplifications) to PL and DIFF. At the same time, Mazur's original infinite process algebra is the heart of the proof. Further, dimension 3 is not really exceptional. Indeed, as Stallings observed, provided the theorem is suitably stated, it holds good in all dimensions. Stallings deals with DIFF only; his proof [60] differs significantly from ours, but one can adapt it to PL and probably to TOP. Finally, its TOP version immediately implies the stated TOP Schoenflies theorem. We can thus claim that the Cantrell-Stallings theorem, as we present it, is an enhancement of the TOP Schoenflies theorem that has exceptional didactic value.

In dimensions  $> 3$ , it is tempting to believe that there is a well-defined notion of CSI for open oriented CAT manifolds with just one end, one that is independent of auxiliary choices in our definition of CSI—notably that of a so-called *flange* (see Section 2) in each summand, or equivalently that of a proper homotopy class of maps of  $[0, \infty)$  to each summand. It has been known since the 1980s [18] that such a proper homotopy class is unique whenever the fundamental group system of connected neighborhoods of infinity is Mittag-Leffler (this means that the system is in a certain sense equivalent to a sequence of group surjections). More recently [18, pages 369–371], it has been established that there are uncountably many such proper homotopy

classes whenever the Mittag-Leffler condition fails; given one of them, all others are classified by the non-null elements of the (first) derived projective limit of the fundamental group system at infinity. This interesting classification does not readily imply that rechoice of flanges can alter the underlying manifold isomorphism type of a CSI sum in the present context; however, we conjecture that it can indeed.

A classification of CAT *multiple* codimension 1 hyperplane embeddings in  $\mathbf{R}^m$ , for  $m \neq 3$ , will be established in Section 9 showing that they are classified by countable simplicial trees with one edge for each hyperplane. This result is called the Multiple Hyperplane Linearization theorem, or MHLT for short (see Theorem 9.2). For TOP and  $m > 3$ , its proof requires the Slab Theorem of Greathouse [24], for which we include a proof, that (inevitably) appeals to the famous Annulus theorem [34]. For dimension  $m = 2$ , we present a bouquet of three quite different proofs of MHLT. First, we explain in detail a hopefully novel proof that uses elementary Morse-theoretic methods to directly classify so-called ‘multirays’ in  $\mathbf{R}^2$  up to ambient isotopy (see Proposition 8.5, Theorem 9.13 and Remark 4.7). These methods may not give the shortest proof. But, on the other hand, we are able to indicate further applications of them, both in dimension 2 and in dimensions  $> 3$ ; see 9.19, 9.20 and 9.21. Second, we show that MHLT for dimension 2 can be reduced to classical results of Schoenflies and K erekjart o which imply a classification of all separable contractible surfaces with nonempty boundary; for this, Section 9 gives an outline, whilst the lecture notes [56] give details. The third and last proof uses an elementary classification of the same surfaces using planar hyperbolic geometry.

The high-dimensional MHLT (our Theorem 9.2) is the hitherto unproved result that brought this article into being! Indeed, the first two authors queried the third concerning an asserted classification for  $m > 3$  in [54, Theorem 10.10, page 117], that is there both unproved and misstated. This simplicial classification is used in [7] to make certain noncompact manifolds real algebraic.

As is often the case with a general notion, particular cases of CSI, sometimes called *end sum*, have already appeared in the literature. Notably, Gompf [21] used end sum for DIFF 4-manifolds homeomorphic to  $\mathbf{R}^4$ , Myers [50] and Tinsley and Wright [62] used end sum for 3-manifolds, and F. Ancel observed (unpublished) in the 1980s that some Davis manifolds [14] appear as end sums (see Remark 2.8 be-

low). The present paper hopefully provides the first general treatment of CSI. However, we give at most fleeting mention of CSI for dimension 2, because, on one hand, its development would be more technical (non-abelian, see Remark 4.7 and [59]), and on the other, its accomplishments are meager.

This paper is organized as follows. Section 2 defines CSI and states its basic properties. Section 3 is a short discussion of certain CAT regular neighborhoods of noncompact submanifolds. Sections 4 and 5 prove the basic properties of CSI. Section 6 uses CSI to prove the Cantrell-Stallings hyperplane unknotting theorem (HLT, Theorem 6.1). Section 7 applies results of Homma and Gluck to TOP rays to derive Cantrell's HLT (Theorem 7.3 for TOP). Section 8 studies proper maps and proper embeddings of multiple copies of  $[0, \infty)$ . Section 9 classifies embeddings of multiple hyperplanes (MHLT, Theorem 9.2). It includes an exposition of Greathouse's Slab Theorem, and in conclusion some possibly novel proofs of the two-dimensional MHLT and related results classifying contractible 2-manifolds with boundary.

We authors believe the best way to assimilate the coming sections is to proceed as we did in writing them; namely, at an early stage, attempt to grasp in outline the proof in Section 6 of the central theorem HLT (Theorem 6.1), and only then fill in the necessary foundational material. Later, pursue some of the interesting side-issues lodged in other sections.

**2. CSI: Connected sum at infinity.** Connected sum at infinity CSI will now be defined for suitably equipped, connected CAT manifolds of the same dimension  $\geq 3$ . (Dimensions  $\leq 2$  seem to lack enough room to make CSI a fruitful notion.) The most common forms of connected sum are the usual connected sum CS and connected sum along boundary CSB; we assume some familiarity with these. All three are derived from disjoint sum by a suitable geometric procedure that produces a new connected CAT manifold. CSI is roughly what happens to manifold interiors under CSB.

Recall that, to ensure well-definition, CS and CSB both require some choices and technology, particularly for TOP. CS requires choice of an embedded disk and appeals to an ambient isotopy classification of them; for TOP, this classification requires the (difficult) stable

homeomorphism theorem (SHT), which will be discussed in Section 9. CSB requires distinguished and oriented boundary disks where the CSB is to take place. Since any CSB operation induces a CS operation of boundaries, it is clear that the extra boundary data for CSB is essential for its well-definition—as dimension 3 already shows. For example, let  $X = S^1 \times D^2$  and  $Y = X - \text{Int } D^3$  where  $D^3$  is a small round disk in  $\text{Int } X$ . The CSB operation on  $X$  and  $Y$  can produce two manifolds with non-homeomorphic boundaries.

The definition of CSI has similar problems, and this imposes the notion of a *flange*, which we define next.

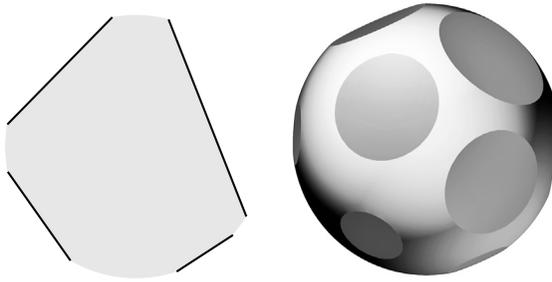
In any CAT, connected, noncompact  $m$ -manifold  $M$ , one can choose a CAT, codimension 0, proper, *oriented* submanifold  $P \subset \text{Int } M$  that is CAT isomorphic to the closed upper half space  $\mathbf{R}_+^m$ . For example,  $P$  can be derived from a suitably defined *regular neighborhood* of a ray  $r$ , where a ray is, by definition, a (proper) CAT embedding of  $[0, \infty)$ . Such a  $P$  with its orientation is called a CSI *flange*, or (for brevity) a *flange*. The pair  $(M, P)$  is called a CSI *pair* or synonymously a *flanged manifold*. Often, a single alphabetical symbol like  $N$  will stand for a flanged manifold; then  $|N|$  will denote the underlying manifold (flange forgotten). Thus, when  $N = (M, P)$ , one has  $|N| := M$ .

In practice, rays and flanges are usually obvious or somehow given by the context, even in dimension 3 where rays can be knotted. For example:

(i) If  $M$  is oriented (or even merely oriented near infinity), it is to be understood that the CSI flange orientation agrees with that of  $M$ —unless this requirement is explicitly waived.

(ii) If  $M$  is a compact manifold with a connected boundary, then  $\text{Int } M$  has a preferred ray up to ambient isotopy; it arises as a fiber of a collaring of  $\partial M$  in  $M$ ; this is because of a well-known collaring uniqueness up to (ambient) isotopy that is valid in all three categories, cf. [36].

(iii) With the data of (ii), suppose  $\partial M$  is oriented. Then the preferred class of rays from (ii) and the isotopy uniqueness of regular neighborhoods (see Section 3) provide a preferred (oriented) flange for  $\text{Int } M$  that is well-defined up to ambient isotopy of  $\text{Int } M$ . On the other hand, if  $\partial M$  is non-orientable, then an ambient isotopy of  $M$  can reverse the orientation of a regular neighborhood in  $M$  of any point of



(a) A 2-dimensional gasket.      (b) A 3-dimensional gasket.

FIGURE 1. Linear gaskets.

$\partial M$ ; hence, in this case also there is an (oriented) flange for  $\text{Int } M$  that is well-defined up to ambient isotopy of  $M$ .

(iv) If  $N$  has dimension  $\leq 3$  and is isomorphic to the interior of a compact manifold with connected boundary, then once again  $N$  has a preferred ray up to isotopy; this is because  $N$  is irreducible near  $\infty$  and irreducible  $h$ -cobordisms of dimension  $\leq 3$  are products with  $[0, 1]$  (see [27]).

A second ingredient for a CSI sum of  $m$ -manifolds will be a so-called *gasket*. The prototypical gasket is a *linear gasket*; this is, by definition, a closed subset of a certain model  $\mathbf{H}^m$  of hyperbolic  $m$ -space whose frontier is a nonempty collection of at most countably many disjoint codimension 1 hyperplanes (see Figure 1). We adopt Felix Klein's projective model of hyperbolic space; in it,  $\mathbf{H}^m$  is the open unit ball in  $\mathbf{R}^m$ , and each codimension 1 hyperbolic hyperplane is by definition a nonempty intersection with  $\mathbf{H}^m$  of an affine linear  $(m - 1)$ -plane in  $\mathbf{R}^m$ . A *gasket* is, by definition, any oriented CAT  $m$ -manifold that is degree +1 CAT isomorphic to a linear gasket.

*Remark 2.1.* A linear gasket is clearly simultaneously an oriented manifold of all three categories. The hyperbolic structure of  $\mathbf{H}^m$  will occasionally be helpful. However, it can be treacherous for PL, since its isometries are not all PL; they are projective linear but mostly not affine linear (not even piecewise). Thus, our mainstay will be the CAT structures inherited from  $\mathbf{R}^m$ .

Consider an indexed set  $\mu_i = (M_i, P_i)$  of CSI pairs of dimension  $m$ , where  $i$  ranges over a nonempty finite or countable index set  $S$ . The CSI

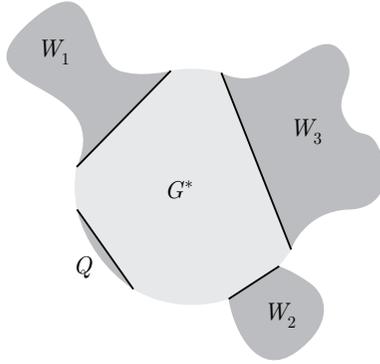


FIGURE 2. CSI operation.

operation yields a CSI pair  $\omega = (W, Q)$  by the following construction (see Figure 2).

Let  $G^*$  be a linear gasket of the same dimension  $m$ , with  $|S| + 1$  boundary components. Each closed component of the complement of  $G^*$  in  $\mathbf{H}^m$  is a CAT flange. We choose one, say  $Q$ , and write  $G$  for the gasket  $G^* \cup Q$ . The flange  $Q$  will become the flange of  $\omega$ .

A pair that is CAT isomorphic to  $(G, Q) := (G^* \cup Q, Q)$  as above will be called a *flanged gasket*. Equivalently, any CSI pair  $(G', Q')$  where  $G'$  and  $G' - \text{Int } Q'$  are both CAT gaskets is by definition a flanged gasket.

$W$  will now be formed by introducing identifications in the disjoint sum:

$$(\dagger) \quad \bigsqcup \{M_i \mid i \in S\} \sqcup G.$$

We index by  $S$  the  $|S|$  components of  $\partial G$ , denoting them by  $H_i, i \in S$ , and choose, for each, a CAT degree +1 embedding  $\theta_i : P_i \rightarrow G^*$  onto an open collar neighborhood of  $H_i$  in  $G^*$ . Now form  $W$  from the disjoint sum  $(\dagger)$  by identifying  $P_i$  to its image in  $G^*$  under  $\theta_i$ . Finally,  $\omega := (W, Q)$  is by definition a CSI *sum* of the CSI pairs  $\mu_i, i \in S$ .

We will call  $G$  and  $G^*$ , respectively, the *coarse gasket* and the *fine gasket* of the CSI sum  $\omega = (W, Q)$ .

*Remark 2.2.* As a topological space,  $W$  is somewhat more simply expressed as the quotient space of the disjoint sum

$$\bigsqcup \{M_i - \text{Int } P_i \mid i \in S\} \sqcup G$$

by the identifications

$$\theta_i|_{\partial P_i} : \partial P_i \longrightarrow H_i.$$

In the PL category, these identifications induce a unique PL manifold structure on  $W$ . But in the DIFF category, the full collarings  $\theta_i$  serve to provide a well-defined differentiable manifold structure on  $W$ .

**Theorem 2.3.** *The CSI of a nonempty but countable (or finite) set of CSI pairs of dimension  $m \geq 3$  enjoys the following properties:*

(1) *From such a set  $(M_i, P_i)$ ,  $i \in S$ , the CSI construction above yields a CSI pair  $(W, Q)$  that is well-defined up to CAT isomorphism. Given a second such construction whose entries are distinguished by primes, a bijection  $\varphi : S \rightarrow S'$ , and, for each  $i \in S$ , an isomorphism of CAT CSI pairs  $\psi_i : (M_i, P_i) \rightarrow (M'_{\varphi(i)}, P'_{\varphi(i)})$ , a CAT isomorphism  $\psi : (W, G, Q) \rightarrow (W', G', Q')$  exists that extends  $\psi_i$  restricted to  $M_i - \text{Int } P_i$  for all  $i \in S$ . Furthermore, this  $\psi$  is degree +1 as a map  $G \rightarrow G'$  and induces an isomorphism of CSI pairs  $(W, Q) \rightarrow (W', Q')$ . Thus, in addition to being well-defined, the CSI operation is commutative.*

(2) *The composite CSI operation is associative.*

(3) *The CSI operation has an identity element  $\varepsilon = (\mathbf{R}^m, \mathbf{R}^m_+)$ , and the infinite CSI product  $\varepsilon\varepsilon\varepsilon \cdots$  of copies of  $\varepsilon$  is isomorphic to  $\varepsilon$ .*

Precise definitions of composite CSI operations and of their associativity are given below in Section 5.

**Notation 2.4.** Theorem 2.3 justifies the following notations for CSI sums. If  $\mathcal{M}$  is a nonempty but countable collection of flanged manifolds, then  $\text{CSI}(\mathcal{M})$  can denote the flanged manifold resulting from the CSI operation applied to these manifolds. And, in case  $\mathcal{M}$  is an ordered sequence  $M_1, M_2, \dots$ , then  $\text{CSI}(M_1, M_2, \dots)$  and  $\text{CSI}(\mathcal{M})$  should be synonymous. An alternative to  $\text{CSI}(M_1, M_2, \dots)$  introduced by Gompf [21] is  $M_1 \natural M_2 \natural \cdots$ .

*Remark 2.5.* In Theorem 2.3, it is already striking that every infinite CSI product yields a well-defined CSI pair (up to isomorphism). Nothing so strong is true for CS or CSB unless artificial limitations are imposed on the infinite connected sum operation. For example, in dimensions  $m \geq 2$ , an infinite CS of any closed, connected, oriented  $m$ -manifold with itself could reasonably be defined so as to have any conceivable end space—to wit, any nonempty compact subset of the Cantor set.

*Remark 2.6.* For  $CAT = \text{DIFF}$  and  $\text{PL}$ , as observed in remarks at the beginning of this section, the interior of a  $CAT$  compact  $m$ -manifold with nonempty connected boundary, has a privileged choice of flange (up to ambient isotopy and orientation reversal). This lets us perceive some near overlap of CSI with the ordinary connected sum CS as follows. Let us suppose that  $M$  is the connected sum  $M_1 \# M_2 \# \dots \# M_k$  of a finite collection  $M_1, \dots, M_k$  of oriented connected closed  $m$ -manifolds, then  $M-(\text{point})$  is  $CAT$  isomorphic, preserving orientation, to the flanged and oriented manifold  $M'_1 \natural M'_2 \natural \dots \natural M'_k$  where  $M'_i$  is the manifold  $M_i-(\text{point})$  with a flange chosen whose orientation agrees with that of  $M_i$ . The reader is left to further explore such relations between CSI and CS.

*Remark 2.7.* The last remark above leads us to simple examples where reversal of a flange orientation changes the underlying proper homotopy type of the CSI of two flanged manifolds.

It is a familiar fact that, if  $M$  is the complex projective plane (of real dimension 4), the ordinary connected sum  $M \# (-M)$  has a signature zero cup product bilinear form on the cohomology group

$$H^2(M \# (-M); \mathbf{Z}) = \mathbf{Z}^2,$$

whilst  $M \# M$  has form of signature  $+2$  (the sign  $+$  becoming  $-$  if we replace  $M$  by  $-M$ ). It follows that  $M \# M$  and  $M \# -M$  are not homotopy equivalent.

Let  $N$  be  $M-(\text{point})$ , the complement of a point in  $M$ , and forget the orientation of  $N$ , but then consider two flanges  $P_+$  and  $P_-$  for  $N$  whose orientations agree with those of  $M$  and  $-M$ , respectively. By Remark 2.6, the CSI of  $(N, P_+)$  and  $(N, P_-)$  is  $(M \# -M)-(point)$

whose Alexandroff one-point compactification is  $(M \# - M)$ . On the other hand, the CSI of  $(N, P_+)$  and  $(N, P_+)$  is  $(M \# M)$ –(point) whose one-point compactification is  $(M \# M)$ . There cannot be a *proper* homotopy equivalence between

$$(M \# - M) - (\text{point}) \quad \text{and} \quad (M \# M) - (\text{point})$$

because its one-point compactification would clearly be a homotopy equivalence between  $M \# - M$  and  $M \# M$ , which does not exist.

*Remark 2.8.* Overlap of CSI and certain CSB sums was observed by Ancel in the 1980s (unpublished). Namely, suppose a noncompact  $n$ -manifold  $W$  is built inductively from a sequence  $M_1, M_2, \dots$  of compact  $n$ -manifolds with nonempty connected boundaries by letting  $N_1 = M_1$ , and letting  $N_{k+1}$  be the CSB of  $N_k$  and  $M_{k+1}$  in such a way that, for each  $k \geq 1$ , a  $j > k$  exists such that  $N_k \subset \text{Int } N_j$ . Then  $W = \cup_k N_k$  is homeomorphic to  $\text{CSI}(\text{Int } M_1, \text{Int } M_2, \dots)$ .

The proof of Theorem 2.3 will mostly be elementary. There is one important exception: the TOP version as presently stated requires the difficult stable homeomorphism theorem (SHT) of [15, 17, 34] to show that any homeomorphism of  $\mathbf{R}^{m-1}$  is isotopic to a linear map. In contrast, for  $\text{CAT}=\text{PL}$  or  $\text{CAT}=\text{DIFF}$ , it is elementary that every  $\text{CAT}$  automorphism of Euclidean space is  $\text{CAT}$  isotopic to a linear map (for PL see [51], and for DIFF see [47, page 34]).

Happily, this dependence on a difficult result can and will be removed. Our tactic is to refine the definition of CSI for TOP *requiring henceforth* (unless the contrary is indicated) that:

- The CSI flange  $P$  in each CSI pair  $(M, P)$  shall carry a preferred DIFF structure making  $P$  DIFF isomorphic to  $\mathbf{R}_+^m$ , and, with respect to such structures, every CSI pair isomorphism shall be DIFF on the flanges.
- Every gasket shall be equipped with a DIFF structure making it DIFF isomorphic to a linear gasket, and all of the identifications made in CSI constructions shall be DIFF identifications with respect to these preferred DIFF structures.

The magical effect of this refined definition is that the proof for DIFF of the basic properties of CSI applies without essential changes to the

TOP category. This is rather obvious if one thinks of TOP CSI as being DIFF where all of the relevant action takes place. Consequently, for many cases of Theorem 2.3, we give little or no proof for the TOP category—leaving the reader to do his own soul searching. Note that the above refinement could equally use PL in place of DIFF.

**3. Regular neighborhoods.** Regular neighborhoods will play a central technical role throughout this article. A short discussion of such CAT neighborhoods, just sufficient for our uses, is given below.

**PL regular neighborhoods.** PL regular neighborhood theory is a major feature of PL topology that is entirely elementary but not always simple. Such a theory was first formulated by Whitehead [64], and then simplified and improved by Zeeman [31, 66] (see also [51]). We need the version of this theory that applies to possibly noncompact PL spaces; it is developed in [53]. We now review some key facts.

Let  $X$  be a closed PL subspace of the PL space  $M$ . Neither is assumed to be compact, connected, nor even a PL manifold. Recall that  $X$  is a subcomplex of some PL triangulation of  $M$  by a locally finite simplicial complex. A regular neighborhood  $N$  of  $X$  can be defined to be a closed  $\varepsilon$ -neighborhood ( $\varepsilon < 0.5$ ) of  $X$  in  $M$  for the barycentric metric of some such triangulation of  $M$ . The frontier of  $N$  in  $M$  is thus PL bicollared in  $M$ .

We quickly recite some familiar facts. Any two regular neighborhoods  $N$  and  $N'$  of  $X$  in  $M$  are ambient isotopic fixing  $X$ . If  $N_0$  is a regular neighborhood that lies in the (topological) interior  $\text{int } N$  of  $N$  in  $M$ , then the triad  $(N - \text{int } N_0; \delta N_0, \delta N)$  is PL isomorphic to the product triad  $\delta N \times ([0, 1]; 0, 1)$  where  $\delta$  indicates the frontier in  $M$ . Thus, if  $N_0$  is contained in  $\text{int } N \cap \text{int } N'$ , and  $U$  is a neighborhood of  $N \cup N'$  in  $M$ , then the ambient isotopy carrying  $N$  to  $N'$  can be the identity on  $N_0$  and on the complement of  $U$ .

We will also use (in some special cases) two less familiar facts, namely Propositions 3.1 and 3.2.

**Proposition 3.1.** *If  $N_i$  is a regular neighborhood of  $X_i$  in  $M_i$  for  $i = 1$  and  $i = 2$ , then  $N_1 \times N_2$  is a regular neighborhood of  $X_1 \times X_2$  in  $M_1 \times M_2$ .  $\square$*

**Proposition 3.2.** *Let  $N$  be a properly embedded  $m$ -submanifold of a PL  $m$ -manifold  $M$  such that  $N \subset \text{Int } M$ , and let  $X$  be a properly embedded PL subspace of  $M$  with  $X \subset N$ . Then, a sufficient condition for  $N$  to be a regular neighborhood of  $X$  in  $M$  is that  $(N, X)$  be PL isomorphic to a pair  $(N', X')$  where  $N'$  is a regular neighborhood of  $X'$  in a PL manifold  $M'$ .  $\square$*

**Proposition 3.3.** *If  $\rho : [0, \infty) \rightarrow \mathbf{R}_+^m$  is a proper linear ray embedding with image  $r$  in  $\text{Int } \mathbf{R}_+^m$ , then  $\mathbf{R}_+^m$  is PL isomorphic fixing  $r$  to a regular neighborhood of  $r$  in  $\mathbf{R}_+^m$ .*

*Proof of Proposition 3.3 from Propositions 3.1 and 3.2.* Adjusting  $r$  by an affine linear automorphism of  $\mathbf{R}_+^m$ , we may assume, without loss of generality, that  $r = 0 \times [2, \infty)$ , where the 0 here denotes the origin of  $\mathbf{R}^{m-1} = \partial\mathbf{R}_+^m$ .

For any real  $\lambda > 0$  and integer  $k > 0$ , let  $B_\lambda^k := [-\lambda, \lambda]^k$ , and let  $B_{<\lambda}^k := (-\lambda, \lambda)^k$ . Since each  $B_\lambda^{m-1}$  is a regular neighborhood of the origin, a PL isomorphism exists for any  $\varepsilon \in (0, 1)$ :

$$\varphi : \mathbf{R}^{m-1} - B_{<\varepsilon}^{m-1} \longrightarrow [\varepsilon, \infty) \times \partial B_\varepsilon^{m-1}$$

extending the canonical identification  $\partial B_\varepsilon^{m-1} \cong \varepsilon \times \partial B_\varepsilon^{m-1}$ . Although  $\varphi$  itself is not canonical, we regard it as an identification.

By Proposition 3.1, the product  $B_1^{m-1} \times [1, \infty)$  is a regular neighborhood of  $r$  in  $\mathbf{R}_+^m$ . Thus, by Proposition 3.2, it certainly will suffice to show that, for some  $\varepsilon \in (0, 1)$ , a PL isomorphism

$$(\dagger) \quad h : \mathbf{R}^{m-1} \times [1, \infty) \longrightarrow B_1^{m-1} \times [1, \infty)$$

exists, fixing  $B_\varepsilon^{m-1} \times [2, \infty)$ .

For  $\varepsilon \in (0, 1)$ , it is an elementary fact about PL 2-manifolds that there is a PL isomorphism:

$$\theta : [\varepsilon, \infty) \times [1, \infty) \longrightarrow [\varepsilon, 1] \times [1, \infty)$$

fixing  $\varepsilon \times [1, \infty)$ . Producing with the identity map of  $\partial B_\varepsilon^{m-1}$ , and then extending by the identity over  $B_\varepsilon^{m-1} \times [1, \infty)$ , we get the required PL isomorphism  $h$  for  $(\dagger)$ .  $\square$

**DIFF regular neighborhoods.** There is a quite general elementary theory of smooth regular neighborhoods in DIFF manifolds. Unfortunately, it involves PL, is fastidious to develop, and currently occupies half of the monograph [29] (see also [6]). We therefore cobble together an ad hoc, but bootstrapping, notion of DIFF regular neighborhood for a DIFF ray  $r$  in a DIFF  $m$ -manifold  $M$  ( $r$  is a proper DIFF embedded copy of  $[0, \infty)$  in  $M$ ). This notion will be derived from the well-known notion of a tube about a submanifold and can be extended to most sorts of DIFF submanifolds.

Let  $p : V(r) \rightarrow r$  be the projection of a DIFF tube about the ray  $r$ .  $V(r)$  is a DIFF submanifold of  $M$  lying in  $\text{Int } M$ . It is a trivial DIFF bundle with projection  $p$ , fiber the unit  $(m-1)$ -disk, orthogonal group, and zero section the inclusion of  $r$ . It is not, however, a neighborhood of  $\partial r = b$ , nor a neighborhood of  $r$  itself. Also,  $V(r)$  has undesirable corners. To obtain an acceptable regular neighborhood of  $r$ , we trim  $V(r)$  and add a cap along the butt end  $p^{-1}(b)$  as follows (see Figure 3). Let  $V'(r) \subset V(r)$  be the subbundle of disks of radius  $1/2$ . In the DIFF manifold with boundary (and corners)  $M_0 = M - \text{Int } V(r)$ , the point  $b = \partial r$  is a boundary point and the disk fiber  $E^{m-1}$  of  $V'$  at  $b$  is a tube about  $b$  in  $\partial M_0$ . A tubular neighborhood  $U(b)$  of  $b$  exists in  $M_0$ . By DIFF tube uniqueness, we can arrange that  $U(b) \cap \partial M_0$  coincides with  $E^{m-1}$ . Further, applying DIFF collaring existence and uniqueness to  $\partial M_0$  in  $M$ , we can arrange that  $T(r) = V'(r) \cup U(b)$  is smooth along  $\partial E^{m-1}$ , and hence is a DIFF submanifold of  $M$  without corners. This  $T(r)$  is, by definition, a DIFF *regular neighborhood* of  $r$  in  $M$ .

For a (proper) DIFF submanifold  $L$ , each component of which is a DIFF ray, we further define a DIFF regular neighborhood to be a DIFF codimension 0 submanifold that is a disjoint union of regular neighborhoods of the component rays of  $L$ .

Ambient DIFF isotopy uniqueness of tubes and collars readily establishes ambient DIFF isotopy uniqueness of such DIFF regular neighborhoods. With some care, the isotopy can be kept fixed outside any open neighborhood of the union of two such regular neighborhoods.

Observe that this definition makes it easy to see that  $\mathbf{R}_+^m$  is a DIFF regular neighborhood of any affine linear ray in  $\mathbf{R}^m$  that lies in the interior of  $\mathbf{R}_+^m$ . Indeed, the corresponding tube about  $r$  can have spherical caps as fibers as shown in Figure 4.

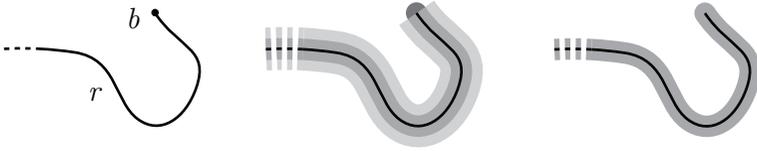


FIGURE 3. A DIFF regular neighborhood of a ray. The light gray (in center figure) indicates a tube about the ray  $r$ , and the dark gray (center and right) indicates the regular neighborhood of  $r$ .

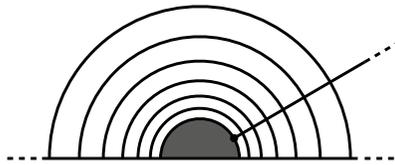


FIGURE 4.  $\mathbf{R}_+^m$  is a DIFF regular neighborhood of an affine linear ray.

*Note.* Proposition 3.3 above is the PL analogue of the previous fact.

**TOP open regular neighborhoods.** There is no simple elementary theory of *closed* TOP regular neighborhoods. This deficiency will be overcome using a simple elementary notion of an *open regular neighborhood* that is adequate for proving the Cantrell-Stallings hyperplane unknotting theorem for TOP using CSI. Incidentally, such open regular neighborhoods could serve in proving the PL and DIFF versions of the hyperplane unknotting theorem, in lieu of the more precise closed CAT regular neighborhood theory.

Let  $W, X, Y$  and  $Z$  be locally compact (but not necessarily compact!) metrizable spaces, where  $Z$  is a closed subset of  $W$ . Consider a proper continuous surjection  $f : X \rightarrow Z$ , and define the *infinite radius mapping cylinder*  $\text{Map}(f)$  to be the quotient of the disjoint union  $X \times [0, \infty) \sqcup Z$  by the relation that identifies  $(x, 0)$  to  $f(x) \in Z$  for all  $x \in X$ . Clearly,  $Z$  is closed in  $\text{Map}(f)$ , and the open subset  $X \times (0, \infty)$  is its complement. For  $\rho > 0$ , we define the *radius  $\rho$  mapping cylinder*  $\text{Map}_\rho(f)$  to be the quotient of  $X \times [0, \rho] \sqcup Z$  in  $\text{Map}(f)$  and also the open one  $\text{Map}_{<\rho}(f)$  to be the quotient of  $X \times [0, \rho) \sqcup Z$  in  $\text{Map}(f)$ .

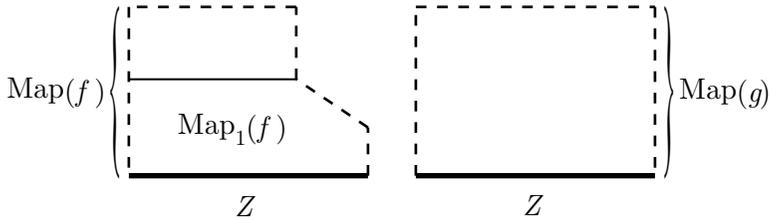


FIGURE 5. Example where  $\text{Map}(f) \subset \text{Map}(g)$ , but  $\text{Map}_1(f)$  is not closed in  $\text{Map}(g)$  and hence is not closed in any space  $W$  containing  $\text{Map}(g)$ . Here,  $\text{Map}(g) = (0, 1) \times [0, \infty)$ ,  $Z = (0, 1) \times 0$  and  $X = (0, 1)$ .

Let  $g : Y \rightarrow Z$  be another such map (same target, but different source). Suppose that  $\text{Map}(f)$  and  $\text{Map}(g)$  are embedded, fixing  $Z$ , as open neighborhoods of  $Z$  in  $W$ . Then, we have the following well-known result, where  $A \Subset B$  for sets in a space  $W$  means that the closure of  $A$  in  $W$  is contained in the interior of  $B$  in  $W$ .

**Theorem 3.4** (Open mapping cylinder neighborhood uniqueness). *If  $\text{Map}_1(f) \Subset \text{Map}(g)$ , then a homeomorphism of  $\text{Map}(f)$  onto  $\text{Map}(g)$  exists that fixes pointwise  $\text{Map}_1(f)$ . Consequently*

$$\text{Map}(g) - \text{Map}_{<1}(f)$$

*is homeomorphic to  $X \times [1, \infty)$  fixing  $X \times 1$ .*

*Remarks 3.5.* (1) Although  $\text{Map}_1(f)$  is clearly closed in  $\text{Map}(f)$ , the conclusion of the theorem is false if  $\text{Map}_1(f)$  is not closed in  $\text{Map}(g)$ , and this may occur even when  $\text{Map}(f) \subset \text{Map}(g)$  as shown in Figure 5. On the other hand, if  $\text{Map}(f) \subset \text{Map}(g)$ , then a self-homeomorphism  $h$  of  $\text{Map}(f)$  exists such that  $h(\text{Map}_1(f))$  is closed in  $W$ .

(2) Even when  $X$  and  $Y$  are both TOP manifolds, the conclusion of this theorem does not imply that  $X$  is homeomorphic to  $Y$ . Further, if they happen to be homeomorphic,  $X \times 1$  is not in general ambient isotopic to  $Y \times 1$  (see [43] and the TOP invariance of simple homotopy type in [35], along with Essay III of [36]).

(3) Theorem 3.4 remains true if  $X, Y$  and  $Z$  are merely Hausdorff and paracompact [57, page 260], but we do not need this generality.

*Proof.* The most appropriate proof to recall here is one using an infinite composition trick that is often called the ‘Eilenberg-Mazur swindle’ (see also [59, 60]). Without loss of generality, we may assume that  $W = \text{Map}(g)$ . After reembedding  $\text{Map}(f)$  and  $\text{Map}(g)$  into  $W$  by suitable topological automorphisms of  $\text{Map}(f)$  and  $\text{Map}(g)$ , respectively, with their supports disjoint from  $\text{Map}_1(f)$ , we can assume that radius 1 and radius 2 mapping cylinders are shuffled as follows

$$(*) \quad \text{Map}_1(f) \Subset \text{Map}_1(g) \Subset \text{Map}_2(f) \Subset \text{Map}_2(g).$$

Here one uses the local compactness and metrizability hypotheses (see [37]).

The triad  $\gamma = (V; X \times 1, Y \times 1)$ , where  $V$  is  $\text{Map}_1(g)$  minus the topological interior of  $\text{Map}_1(f)$ , can be regarded as a cobordism from  $X$  to  $Y$  (see [46]). In this context, cobordism means that the two subspaces of each triad are identified in the obvious way to  $X$  or to  $Y$ . Cobordism isomorphism (indicated by  $\cong$ ) means triad homeomorphism respecting these identifications.

The relations  $(*)$  show that  $\gamma$  has an inverse  $\gamma' = (V'; Y \times 1, X \times 2)$  viewed as a cobordism from  $Y$  to  $X$ , where  $V'$  is  $\text{Map}_2(f)$  minus the topological interior of  $\text{Map}_1(g)$  (see Figure 6). In other words, the end to end cobordism composition  $\gamma \cdot \gamma'$  is topologically the product cobordism  $\varepsilon_X$  on  $X$ , written  $\gamma \cdot \gamma' \cong \varepsilon_X$ . Similarly,  $\gamma'$  has an inverse  $\gamma'' = (V''; X \times 2, Y \times 2)$ , where  $V''$  is  $\text{Map}_2(g)$  minus the topological interior of  $\text{Map}_2(f)$ , written  $\gamma' \cdot \gamma'' \cong \varepsilon_Y$ . Using an obvious associativity, we see that  $\gamma$  and  $\gamma''$  are isomorphic cobordisms

$$\gamma \cong \gamma \cdot \varepsilon_Y \cong \gamma \cdot (\gamma' \cdot \gamma'') \cong (\gamma \cdot \gamma') \cdot \gamma'' \cong \varepsilon_X \cdot \gamma'' \cong \gamma''.$$

In particular,  $\gamma' \cdot \gamma \cong \varepsilon_Y$ .

$\text{Map}(f)$  minus the interior of  $\text{Map}_1(f)$  is (the body of) the infinite cobordism composition  $\varepsilon_X \cdot \varepsilon_X \cdot \varepsilon_X \cdots$ , while  $\text{Map}(g)$  minus the interior of  $\text{Map}_1(f)$  is the infinite composition  $\gamma \cdot \varepsilon_Y \cdot \varepsilon_Y \cdot \varepsilon_Y \cdots$ . But, these are the same by the infinite product swindle, again using associativity

$$\begin{aligned} \gamma \cdot \varepsilon_Y \cdot \varepsilon_Y \cdot \varepsilon_Y \cdots &\cong \gamma \cdot (\gamma' \cdot \gamma) \cdot (\gamma' \cdot \gamma) \cdot (\gamma' \cdot \gamma) \cdots \\ &\cong (\gamma \cdot \gamma') \cdot (\gamma \cdot \gamma') \cdot (\gamma \cdot \gamma') \cdots \cong \varepsilon_X \cdot \varepsilon_X \cdot \varepsilon_X \cdots. \end{aligned}$$

This completes the proof of the theorem. □

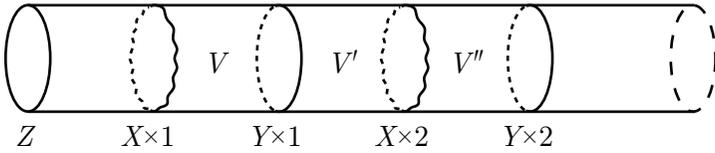


FIGURE 6. Shuffled mapping cylinders.

**4. Radial ray and linear gasket uniqueness.** In this section, CAT will mean either PL or DIFF. We begin with a ray unknotting lemma for radial rays in  $\mathbf{H}^m$ . Let  $L \cong \mathbf{Z}_+ \times [0, \infty)$  be a proper CAT embedded submanifold of  $\mathbf{H}^m$  so that all rays  $r_i = i \times [0, \infty)$  are *radial*, i.e., each ray is contained in a line through the origin in  $\mathbf{R}^m \supset \mathbf{H}^m$ , and is disjoint from the origin. In what follows, lengths come from the standard Euclidean metric on  $\mathbf{R}^m$ . For each  $i \in \mathbf{Z}_+$ , let  $d_i$  denote the distance from the origin in  $\mathbf{R}^m$  to the initial point of  $r_i$  parameterized by  $i \times 0$ , and let  $p_i$  denote the limit point of  $r_i$  in  $S^{m-1} = \partial \overline{\mathbf{H}}^m$ , where  $\overline{\mathbf{H}}^m$  is the unit ball  $B^m$  that is the closure of  $\mathbf{H}^m$  in  $\mathbf{R}^m$ .

Let  $L'$  be another such submanifold, and define  $r'_i, d'_i$  and  $p'_i$  in the same way. Also, let  $f : L \rightarrow L'$  be a CAT isomorphism. Notice that both sequences  $d_i$  and  $d'_i$  converge to 1 as  $i \rightarrow \infty$  since  $L$  and  $L'$  are properly embedded.

$S(t_1, t_2)$ , with  $0 < t_1 < t_2 \leq 1$ , will denote the thickened sphere of points  $x \in \mathbf{H}^m$  such that  $t_1 \leq \|x\| \leq t_2$ .

**Lemma 4.1** (Radial ray uniqueness). *With the above data, suppose  $m \geq 3$ . Then there is a CAT ambient isotopy  $h_t$  of  $\mathbf{H}^m$ ,  $0 \leq t \leq 1$ , such that  $h_0$  is the identity and  $h_1|_L = f$ .*

*Remark 4.2.* This ambient isotopy of  $\mathbf{H}^m$  cannot in general extend to an ambient isotopy of the ball  $\overline{\mathbf{H}}^m$  since the accumulation points in  $\partial \overline{\mathbf{H}}^m$  of  $L$  would then be homeomorphic to those of  $L'$ . On the other hand,  $p_i$  can be an arbitrary sequence of distinct points in  $\partial \overline{\mathbf{H}}^m$ ; thus, its set of accumulation points in  $\partial \overline{\mathbf{H}}^m$  may be any nonempty compact subset.

*Proof of Lemma 4.1 for DIFF.* Reindex the rays  $r_i$  so that  $f(r_i) = r'_i$ . Since any DIFF automorphism of  $[0, \infty)$  is isotopic to the identity, it

will suffice to construct  $h_t$  as above so that  $h_1(r_i) = r'_i$ . Reindexing rays, we can assume that  $d_i \leq d_{i+1}$  for  $i \in \mathbf{Z}_+$ . It is elementary that  $d_i \rightarrow 1$  as  $i \rightarrow \infty$ .

A preliminary ambient isotopy sets the stage. Shrink the rays  $r_i$  radially towards their limit points  $p_i$  so that  $d_i \geq d'_i$  (while maintaining  $d_i \leq d_{i+1}$ ). This is straightforward using a regular neighborhood of  $L$  in  $\mathbf{H}^m$ .

Now, choose a DIFF simple path  $\alpha_1$  in  $S^{m-1}$  from  $p_1$  to  $p'_1$ . This path obviously permits construction of an isotopy of  $S^{m-1}$  supported near the path and taking  $p_1$  to  $p'_1$ . Extending this isotopy radially gives an ambient isotopy of  $S(d_1, 1)$  taking the ray  $r_1$  to a subset of  $r'_1$  (recall, we arranged that  $d_i \geq d'_i$ ). This ambient isotopy of  $S(d_1, 1)$  extends naturally to one of  $\overline{\mathbf{H}}^m$  fixing the ball of radius  $d_1 - \varepsilon_1$  for any small  $\varepsilon_1 > 0$ . At the end of this isotopy, any ray  $r_i$ ,  $i \geq 2$ , that moved has image another radial ray of the same length which (abusing language) we still refer to as  $r_i$  with endpoint  $p_i$ .

Next, similarly form an isotopy of  $S^{m-1}$  moving  $p_2$  to  $p'_2$  and having support missing  $p'_1$ . Extending radially to  $S(d_2, 1)$  we get an ambient isotopy (fixing  $r'_1 \supset r_1$ ) taking  $r_2$  to a subset of  $r'_2$ .

Inductively form an isotopy of  $S^{m-1}$  moving  $p_i$  to  $p'_i$  and with support disjoint from  $p'_j$ , for  $1 \leq j \leq i - 1$ . Extend as before to get an ambient isotopy of  $\overline{\mathbf{H}}^m$  with support in  $S(d_i - \varepsilon_i, 1)$  taking  $r_i$  to a subset of  $r'_i$  while fixing  $r'_j \supset r_j$ , for  $1 \leq j \leq i - 1$ . Here,  $\varepsilon_i$  lies in  $(0, d_i)$  and  $\varepsilon_i \rightarrow 0$  as  $i \rightarrow \infty$ . Also, since  $d_i \rightarrow 1$  as  $i \rightarrow \infty$ , the points in any compact set in  $\mathbf{H}^m$  are moved at most finitely many times. Hence, the time interval composition of all of these ambient isotopies provides a well-defined ambient isotopy of  $\mathbf{H}^m$  (but usually not one of  $\overline{\mathbf{H}}^m$ ). We now have  $r_i \subset r'_i$  for all  $i \in \mathbf{Z}_+$ . A final ambient isotopy stretches each  $r_i$  so that  $d_i = d'_i$ , finishing the proof for DIFF.  $\square$

*Proof of Lemma 4.1 for PL.* Make a preliminary PL identification  $\Theta$  of the thickened standard PL  $(m - 1)$ -sphere  $\Sigma^{m-1} \times (0, 1)$  to the complement of the origin  $\mathbf{H}^m - 0$  in such a way that each component of  $L$  and of  $L'$  lies in a modified ray

$$\Theta((\text{point}) \times (0, 1)) \subset \Theta(\Sigma^{m-1} \times (0, 1)) = \mathbf{H}^m - 0.$$

Now, imitate the DIFF proof.  $\square$

*Remark 4.3.* There is no such PL identification  $\Theta$  that sends every ray of the form  $((\text{point}) \times (0, 1)) \subset \Sigma^{m-1} \times (0, 1)$  to a radial ray in  $\mathbf{H}^m - 0$ , not even when  $m = 2$ . This is a corollary of the observation that the point preimages under any linear surjection  $\mathbf{R}^m \rightarrow \mathbf{R}^{m-1}$  are the set of all lines in  $\mathbf{R}^m$  parallel to the kernel line. Thus, the construction of  $\Theta$  must be adapted to  $L$  and  $L'$ , for example, by using a well-chosen triangulation in which  $L$  and  $L'$  are 1-subcomplexes.

Combined with PL and DIFF regular neighborhood theory (see Section 3), the above radial ray uniqueness lemma (Lemma 4.1) will let us prove a linear gasket uniqueness lemma that we now formulate. Adopting the context and terminology established for Lemma 4.1, fix a category CAT to be PL or DIFF. Let  $G$  be a linear gasket of dimension  $m \geq 3$ , i.e., a submanifold of  $\mathbf{H}^m$  bounded by countably many disjoint hyperbolic hyperplanes  $H_i, i \in \mathbf{Z}_+$ . Let  $G'$  be another such gasket of dimension  $m$  and distinguish corresponding subsets by primes.

**Lemma 4.4** (Linear gasket uniqueness). *Given the data above, there is a CAT ambient isotopy  $g_t$  of  $\mathbf{H}^m, 0 \leq t \leq 1$ , so that  $g_0 = \text{id}|_{\mathbf{H}^m}, g_1(G) = G',$  and  $g_1(H_i) = H'_i$  for all  $i \in \mathbf{Z}_+$ .*

**Corollary 4.5.** *If  $G$  and  $G'$  are gaskets and  $f : \partial G \rightarrow \partial G'$  is a degree +1 CAT isomorphism of their boundaries, then  $f$  extends to a CAT isomorphism  $F : G \rightarrow G'$ .  $\square$*

*Proof of Lemma 4.4.* Without loss of generality, we assume that  $G$  and  $G'$  are linear gaskets in  $\mathbf{H}^m$ . The gasket  $G$  determines a canonical CAT submanifold  $L \cong \mathbf{Z}_+ \times [0, \infty)$  of  $\mathbf{H}^m$  as follows: each hyperplane boundary component  $H_i, i \in \mathbf{Z}_+$ , of the gasket  $G$  defines a proper radial ray  $r_i$  in  $\mathbf{H}^m$ , namely, the one orthogonal to  $H_i$ , having endpoint the point of  $H_i$  closest to the origin in  $\mathbf{R}^m$ , and extending outwards from  $G$ . The union of these rays is defined to be  $L$ . For each  $H_i$ , let  $V_i$  denote the closed complementary component of  $\mathbf{H}^m - \text{Int } G$  with boundary  $H_i$ . If  $r$  is any radial ray in  $\mathbf{H}^m$ , then let  $s$  denote the radial ray obtained from  $r$  by shrinking it outwards radially to be half as long (for the Euclidean metric). Each  $V_i$  is isomorphic to the closed upper half space  $\mathbf{R}_+^m$  and is a CAT regular neighborhood of  $s_i$  as we have

observed in Section 3. Similarly, the gasket  $G'$  canonically determines closed complementary components  $V'_i$ , rays  $r'_i$ , and shortened rays  $s'_i$ .

After a preliminary isotopy provided by Lemma 4.1 (radial ray uniqueness), we may assume  $r_i = r'_i$  for all  $i \in \mathbf{Z}_+$ , and hence  $s_i = s'_i$  for all  $i \in \mathbf{Z}_+$ . By CAT regular neighborhood ambient uniqueness, we may now ambiently isotop  $V_i$  to  $V'_i$  for all  $i$  (simultaneously), completing the proof.  $\square$

*Remark 4.6.* Lemmas 4.1 and 4.4 also hold for a finite index set in place of  $\mathbf{Z}_+$ . One can deduce this from the case of  $\mathbf{Z}_+$ . Or, one can note that the same proofs apply.

*Remark 4.7.* As it is stated, Lemma 4.1 (radial ray uniqueness) fails in dimension 2, even for three rays. Any set of distinct radial rays in  $\mathbf{H}^2$  obviously inherits a natural cyclic order from that of their limit points on the circle  $S^1$ . The proof of Lemma 4.1 actually shows that such a collection of rays is determined up to ambient isotopy of  $\mathbf{H}^2$  by the isomorphism class of its cyclic ordering. There are many such classes when the number of rays is infinite. For example, the number of rays with no immediate successor (or predecessor) is then an invariant. In fact, there are uncountably many such classes. We are confident that, taking account of this natural ray order, one can nevertheless define an associative CSI operation for 2-manifolds. It is non-commutative in general for 2-manifolds with boundary.

## 5. Proof of Theorem 2.3: Basic properties of CSI.

**Proof of property (1). Well-definition and commutativity of CSI.** Let CAT be PL or DIFF. Recall that, with the data introduced for the statement of property (1) of Theorem 2.3 above, we are seeking a certain sort of CAT isomorphism of triples  $\psi : (W, G, Q) \rightarrow (W', G', Q')$ . On the closed complements of the gasket interiors, this  $\psi$  is rigidly prescribed by the data; call this  $\psi_0 : W - \text{Int } G \rightarrow W' - \text{Int } G'$ . This  $\psi_0$  has degree +1 as a map  $\partial G \rightarrow \partial G'$ . Further, the  $\psi$  we seek is prescribed up to isotopy on  $Q$  as a degree +1 isomorphism  $Q \rightarrow Q'$ . Thus, denoting by  $H$  and  $H'$  the fine gaskets  $G - \text{Int } Q$  and  $G' - \text{Int } Q'$ , it suffices to extend  $\psi| : \partial H \rightarrow \partial H'$  to a CAT degree +1 isomorphism  $H \rightarrow H'$  of the fine gaskets. This extension exists by Corollary 4.5.  $\square$

We explain the notion of a *countable indexed set*  $\mathcal{M}$  of flanged manifolds. It consists of a set  $I$  that is finite or countably infinite, and a map of  $I$  into the class of flanged  $m$ -manifolds. The set  $I$  is called the *index set* and, in what follows, will always be a subset of  $\mathbf{N}$  or of  $\mathbf{N}^2$ . If we write  $\mathcal{M} = \{M_i \mid i \in I\}$ , then the flanged manifold corresponding to  $i \in I$  is  $M_i$ . It is not always required that  $M_i$  and  $M_j$  be disjoint or even distinct when  $i \neq j$  in  $I$ . Thus, one can also similarly define an indexed set in any class—in place of the class of flanged manifolds—for example, in the class of CAT isomorphism classes of flanged manifolds.

**Composite CSI operations and associativity.** To elucidate associativity, we must make its meaning more precise. Our proof of the Cantrell-Stallings hyperplane unknotting theorem uses only a simple (but infinite) associativity which is expressible in traditional algebraic notation. But, the CSI operation enjoys a natural associativity that is at once more general and equally straightforward to establish. Some tree combinatorics will be involved. More specifically, we introduce what we call a *tree of flanged gaskets*. The category CAT in which we work here is again PL or DIFF, and the manifold dimension  $m$  will be  $\geq 3$ .

A *rooted tree* will mean a countable simplicial tree (not necessarily locally finite) that has a distinguished vertex  $v_0$  called the *root*. In such a tree, there is a natural orientation of the edges. Indeed, from each vertex  $v \neq v_0$  there is a unique oriented edge  $vv'$  joining  $v$  to a vertex  $v'$  strictly nearer to the root vertex in the obvious simplicial path metric.

A *tree of  $m$ -dimensional flanged gaskets* is a rooted abstract tree  $\mathcal{G}$  whose vertices and edges are given as follows:

(1) The vertex set of  $\mathcal{G}$  is a finite or countable indexed set  $\{G_i \mid i \in I\}$  of disjoint  $m$ -dimensional flanged gaskets. The flange of  $G_i$  is denoted  $F_i$  and the root vertex of  $\mathcal{G}$  is denoted  $G_0$ . Furthermore, the boundary components of  $G_i$  are indexed as  $H_{i,j}$ ,  $j \in J_i$ .

(2) There is a unique oriented edge of  $\mathcal{G}$  joining any vertex  $G_i \neq G_0$  to the unique adjacent vertex (flanged gasket)  $G_{i'}$  that is nearer to  $G_0$ . This edge is presented as an ordered pair  $(G_i, H_{i',j})$  where, as the notation indicates,  $H_{i',j}$  is one of the indexed boundary components of  $G_{i'}$ . Each boundary component of the disjoint sum  $|G| = \sqcup_i |G_i|$  is required to occur in *at most* one edge of  $\mathcal{G}$ .

By the following gluing process,  $\mathcal{G}$  determines a CAT *composite flanged gasket* denoted by  $\|\mathcal{G}\|$ . In the disjoint sum  $|G| = \sqcup_i |G_i|$ , make these identifications: for each edge  $(G_i, H_{i',j})$  of  $\mathcal{G}$ , identify the flange  $F_i$  of  $G_i$  to a small open collar of  $H_{i',j}$  in  $|G_{i'}|$  by an orientation preserving CAT isomorphism  $\theta_{i,i'}$ . Here ‘small’ should mean inside a prescribed open collar neighborhood of the boundary  $\partial(G_{i'} - F_{i'})$  of the fine gasket of  $G_{i'}$ , so that the flanges identified into  $G_{i'}$  obviously do not intersect. Since degree determines  $\theta_{i,i'}$  up to isotopy,  $\|\mathcal{G}\|$  is determined up to CAT isomorphism that is the identity outside of an arbitrarily small bicollar neighborhood in  $\|\mathcal{G}\|$  of the identified boundary components  $\partial F_i = H_{i',j}$ .

**Lemma 5.1.** *With the above definitions, the pair  $(\|\mathcal{G}\|, F_0)$  is a flanged gasket.*

The proof of Lemma 5.1 will come after we complete the definition of a composite CSI operation based on  $\mathcal{G}$ .

For each  $i \in I$ , consider the set  $J_i^+ \subset J_i$  of those  $j \in J_i$  (if any) such that, for no  $k \in I$ , an edge  $(G_k, H_{i,j})$  exists. By the construction of  $\|\mathcal{G}\|$ , its boundary  $\partial\|\mathcal{G}\|$  is a disjoint sum

$$\bigsqcup \{H_{i,j} \mid i \in I \text{ and } j \in J_i^+\}.$$

By definition, a *composite CSI operation* according to the rooted tree  $\mathcal{G}$  of flanged gaskets as above involves an indexed set of flanged  $m$ -manifolds to be ‘summed’

$$\{M_{i,j} \mid i \in I \text{ and } j \in J_i^+\}.$$

The corresponding ‘sum’ is the flanged manifold (flanged by  $F_0$ ) obtained by gluing the flange of each such  $M_{i,j}$  by a degree +1 isomorphism to a small open collar of  $H_{i,j}$  in  $\|\mathcal{G}\|$  (clearly this open collar may be chosen in  $|G_i|$ ).

*Proof of Lemma 5.1.* With the notations established above, it suffices to prove that the flanged manifold  $\|\mathcal{G}\|$  is a flanged gasket. This is immediate from the following more primitive lemma (which will be reused in Section 9 in our proof of the MHLT (Theorem 9.2)).  $\square$

**Lemma 5.2.** *For CAT=DIFF or PL, suppose that an oriented CAT  $m$ -manifold  $X$  is a finite or countable union of CAT gaskets  $G_i$ ,  $i \in I$ ,*

any two of which are either disjoint or intersect in a single boundary component of each. Suppose also that the nerve of the closed cover  $\{G_i \mid i \in I\}$  of  $X$  is a simplicial tree  $\mathcal{T}$ . Then  $X$  is a CAT gasket.

*Proof of Lemma 5.2.* Without loss of generality, we assume  $I$  is  $\mathbf{N}$  or a finite initial segment of  $\mathbf{N}$ . Reindexing the  $G_i$ , we can arrange that, for all  $i \geq 0$ , the gasket  $G_{i+1}$  is adjacent in  $X$  to the connected block

$$X_i := G_0 \cup G_1 \cup \cdots \cup G_i.$$

By definition,  $G_0 = X_0$  can be degree  $+1$  embedded in  $\mathbf{H}^m$  with frontier made up of hyperbolic hyperplanes.

Suppose inductively that  $\phi_i : X_i \rightarrow \mathbf{H}^m$  is such an embedding for some  $i \geq 0$ . Write  $X'_i$  for  $\phi_i(X_i)$ , write  $H_i$  for the boundary component of  $X_i$  that is shared with  $G_{i+1}$  and write  $H'_i$  for the hyperbolic hyperplane  $\phi_i(H_i)$ . We will extend this embedding  $\phi_i$  to one of  $X_{i+1} = X_i \cup G_{i+1}$ .

Let  $Y_i^+$  be the closed halfspace in  $\mathbf{H}^m$  bounded by  $H'_i$  that does not intersect  $\text{Int } X'_i$ . In  $\text{Int } Y_i^+$  choose as many disjoint halfspaces (each bounded by a hyperbolic hyperplane) as  $G_{i+1}$  has boundary components disjoint from  $X_i$ ; then delete the interiors of those halfspaces from  $Y_i^+$ . With the intent to assure that the ultimate embedding of  $X$  will be proper, we can and do

(\*) choose these halfspaces within the  $1/(i + 1)$  neighborhood of the frontier sphere  $S^{m-1}$  of  $\mathbf{H}^m$  in  $\mathbf{R}^m$  (for the Euclidean distance of  $\mathbf{R}^m$ ).

The result is a linear gasket  $G'_{i+1}$  in  $\mathbf{H}^m$  adjacent to  $X'_i$ , more precisely  $X'_i \cap G'_{i+1} = H'_i$ . By Corollary 4.5 concerning CAT uniqueness of linear gaskets, there is a CAT isomorphism  $G_{i+1} \rightarrow G'_{i+1}$  agreeing with  $\phi_i$  on  $H_i$  and thus extending  $\phi_i|_{H_i}$  to a CAT embedding  $\psi_{i+1}$  of  $G_{i+1}$  onto a linear gasket in  $\mathbf{H}^m$ . Then  $\phi_i$  and  $\psi_{i+1}$  together define an injective CAT map  $X_{i+1} \rightarrow \mathbf{H}^m$  that is clearly proper. For CAT=PL, this injective map induces a PL isomorphism with its image. For CAT=DIFF, this is likewise true after modification of  $\psi_{i+1}$  on a small collar neighborhood of  $H_i$  in  $G_{i+1}$  (see [46]).

This completes the induction defining  $\phi_i$  for  $i \in I$ . The inductively imposed condition (\*) assures that:

(\*\*) For each  $i > 0$ , the frontier  $\partial G'_i$  lies in the  $1/i$  neighborhood of  $S^{m-1}$ .

Hence  $G'_i$  either contains the ball about the origin of Euclidean radius  $1 - (1/i)$  or else it lies outside that ball. Since the sets  $\text{Int } G'_i$  are disjoint, it follows that, for all large  $i$ ,  $G'_i$  lies outside the ball of radius  $1 - (1/i)$ . Since  $\mathbf{H}^m$  is the open ball of radius 1 in  $\mathbf{R}^m$ , we conclude that:

(\*\*\*) The sets  $G'_i$  converge toward Alexandroff's infinity in  $\mathbf{H}^m$ .

Together, the  $\phi_i$  clearly define an injective CAT map  $\phi : X \rightarrow \mathbf{H}^m$ . The condition (\*\*\*) proves that  $\phi$  is proper and thus a CAT embedding onto a linear gasket  $X'$ .  $\square$

*Remark 5.3.* In the proof of Lemma 5.2, if the conditions (\*) and (\*\*) are not imposed and the tree  $\mathcal{T}$  contains an infinitely long embedded path, then the map  $\phi : X \rightarrow \mathbf{H}^m$  may not be proper. But the closure of  $\phi(X)$  always seems to be a linear gasket.

**Proof of property (2): Associativity of CSI operations.** Here we state explicitly, and prove, the associativity properties of CSI as promised in property (2) of Theorem 2.3. We then deduce two basic corollaries.

By Lemma 5.1 and the above definition of composite CSI operation, we immediately get:

**Theorem 5.4** (First associativity theorem). *Any fixed composite CSI operation on a finite or countably infinite set of disjoint flanged CAT  $m$ -manifold summands,  $m \geq 3$ , is isomorphic to a (normal) CSI sum of the same flanged manifolds. Thus the flanged manifold resulting from this composite CSI operation depends (up to CAT isomorphism of flanged manifolds) only on the disjoint sum of the flanged manifold summands.*  $\square$

This quickly implies the

**Theorem 5.5** (Second associativity theorem). *Consider a nonempty sequence  $M_i$  (finite or infinite) of disjoint flanged  $m$ -manifolds,  $m \geq 3$ , where each  $M_i$  is itself a CSI sum of a sequence (finite or infinite) of disjoint flanged manifolds  $M_{i,j}$ . Then, writing  $\mathcal{M}$  for the set  $\{M_i\}$  and  $\mathcal{M}'$  for the set  $\{M_{i,j}\}$ , there is a CAT isomorphism of flanged CSI sums:*

$$\text{CSI}(\mathcal{M}) \cong \text{CSI}(\mathcal{M}').$$

*Proof.* Examine the defining construction for  $\text{CSI}(\mathcal{M})$ , which uses a flanged gasket with  $|\mathcal{M}|$  boundary components. In it, replace each summand  $M_i$  by a copy of  $\text{CSI}(\mathcal{M}_i)$  where  $\mathcal{M}_i := \sqcup_j \{M_{i,j}\}$ . This reveals that  $\text{CSI}(\mathcal{M})$  is isomorphic to a composite CSI sum with summands  $\mathcal{M}'$ . Hence, the first associativity theorem tells us that  $\text{CSI}(\mathcal{M}) \cong \text{CSI}(\mathcal{M}')$ .  $\square$

**Corollary 5.6.** *Let  $\alpha, \beta$  and  $\gamma$  be flanged  $m$ -manifolds,  $m \geq 3$ . Then one has a CAT isomorphism of flanged manifolds  $(\alpha\beta)\gamma \cong \alpha(\beta\gamma)$ .*

This is the usual formulation of associativity for any binary operation. The parentheses in this example and the next serve to indicate order of CSI summation. The expression  $(\alpha\beta)$  indicates the flanged manifold for which we have mentioned the alternative notations  $\text{CSI}(\alpha, \beta)$  and  $(\alpha \natural \beta)$ .

*Proof.* Two applications of the second associativity theorem above give the two isomorphisms:

$$(\alpha\beta)\gamma \cong \alpha\beta\gamma \cong \alpha(\beta\gamma). \quad \square$$

The next corollary will be used in proving the HLT (Theorem 6.1).

**Corollary 5.7.** *Let the symbols  $a, b, c, \dots$  of an infinite alphabet stand for CAT flanged  $m$ -manifolds,  $m \geq 3$ . Then one has a CAT isomorphism of infinite CSI sums of flanged manifolds:*

$$(\dagger) \quad (ab)(cd)(ef)(gh) \cdots \cong a(bc)(de)(fg) \cdots .$$

*Proof.* Applying the second associativity theorem to the left hand side of  $(\dagger)$ , one gets the isomorphism of CSI sums:

$$(ab)(cd)(ef)(gh) \cdots \cong abcdefgh \cdots .$$

Similarly,

$$a(bc)(de)(fg) \cdots \cong abcdefgh \cdots . \quad \square$$

**Proof of property (3): Identity element.** The easy verifications that the CSI identity is  $\varepsilon = (\mathbf{R}^m, \mathbf{R}_+^m)$  and  $\varepsilon \cong \varepsilon\varepsilon\varepsilon\cdots$  are left to the reader.  $\square$

The proof of the three basic properties of CSI (Theorem 2.3) is complete.  $\square$

**6. CSI proves the Cantrell-Stallings hyperplane unknotting theorem.** The machinery developed thus far suffices to prove the following important hyperplane unknotting theorem [9, 60]. Given a manifold  $M$  CAT isomorphic to some  $\mathbf{R}^k$ , we say a CAT ray  $r$  embedded in  $M$  is *unknotted* in  $M$  if there is a CAT isomorphism  $f : \mathbf{R}^k \rightarrow M$  such that  $f^{-1}(r)$  is linear in  $\mathbf{R}^k$ .

**Theorem 6.1** (Hyperplane linearization theorem, HLT). *Consider a codimension 1 and CAT proper submanifold  $N$  of  $\mathbf{R}^m$ ,  $m \geq 2$ , that is CAT isomorphic to  $\mathbf{R}^{m-1}$ . Assume that there is a ray  $r$  in  $N$  that is unknotted both in  $N$  and in  $\mathbf{R}^m$ . Then,  $N$  is itself unknotted in the sense that  $g(N)$  is linear for some CAT automorphism  $g$  of  $\mathbf{R}^m$ .*

*Remark 6.2.* The ray unknotting hypothesis facilitates our CSI based proof for  $m \geq 3$ . The next section shows it is superfluous if  $m > 3$ .

*Remark 6.3.* Dimension 2 is special in that, not only is the ray unknotting hypothesis unnecessary, but in the case of TOP the abiding assumption of local flatness is redundant by the classical Schoenflies theorem (see [48, 55]).

*Proof of the HLT* (Theorem 6.1) *for  $m = 2$ .* This is known by classical methods that are explained in [55]. Some details follow.

Case CAT=TOP. One-point (Alexandroff) compactify the pair  $(\mathbf{R}^2, N)$  to produce a pair  $(S^2, \widehat{N})$ . The (difficult) classical Schoenflies theorem tells us  $(S^2, \widehat{N})$  is homeomorphic to the standard pair  $(S^2, S^1)$ . From this, it follows that, upon deleting the added point  $\infty$ , the pair  $(\mathbf{R}^2, N)$  is homeomorphic to  $(\mathbf{R}^2, \mathbf{R}^1 \times 0)$ .

Case CAT=PL. Proceed similarly but use the “almost PL Schoenflies theorem” (APLST) of Sections 5 and 7 of [55] (see also Remark 6.4 below) to conclude that  $(S^2, \widehat{N})$  is homeomorphic to  $(S^2, S^1)$  by a homeomorphism that is PL except at  $\infty$ . Upon deleting  $\infty$  we get the desired PL isomorphism between  $(\mathbf{R}^2, N)$  and  $(\mathbf{R}^2, \mathbf{R}^1 \times 0)$ .

Case  $CAT=DIFF$ . It is possible to imitate the above proof for PL. Alternatively, embedded Morse theory offers an interesting proof that is described in Remarks 9.19 and 9.20 following the proof of the two-dimensional multiray radialization theorem (MRT, Theorem 9.13) in Section 9.  $\square$

*Remark 6.4* (On overlapping two-dimensional results and techniques). Fortunately, one overlap simplifies: in dimension 2, one can usually shift results, at the statement level, between any two of the three categories DIFF, PL and TOP by appealing to what can be called “2-Hauptvermutung” theorems, for which good references are [48], or [55, Section 9].

Another simplification comes from the coincidence of these three seemingly different properties for connected noncompact surfaces with all boundary components noncompact: irreducibility, planarity, and contractibility. A proof will be given as Proposition 9.24.

On the other hand, in dimension 2, there is a somewhat confusing wealth of techniques and names for them. We now illustrate for the present article.

What is called the “Irreducible PL Surface Classification Theorem” (PLCT) in [55, Section 7] is a direct PL classification, using the very simple PL Schoenflies theorem, for all PL connected noncompact planar surfaces with finitely many boundary components all noncompact. This PLCT was used in [55] to prove the classical Schoenflies theorems that are used in the proofs of 2-HLT just given. Also, PLCT clearly *directly* implies the PL case of 2-HLT.

Serious overlap of techniques is going to appear when we attack the two-dimensional multiple hyperplane linearization theorem (2-MHLT) in Section 9. The PLCT just mentioned will turn out to be synonymous with the case for *finitely many* boundary components of the two-dimensional “Gasket Recognition Theorem” (2-GRT), see Corollary 9.3 below; this 2-GRT generalizes PLCT in that it allows an infinite number of boundary components. Towards the end of Section 9, we will observe that 2-GRT is equivalent to a classification of all contractible 2-manifolds, and we will ultimately give three amazingly different proofs of it, which respectively focus on embedded Morse theory, end theory and hyperbolic geometry.

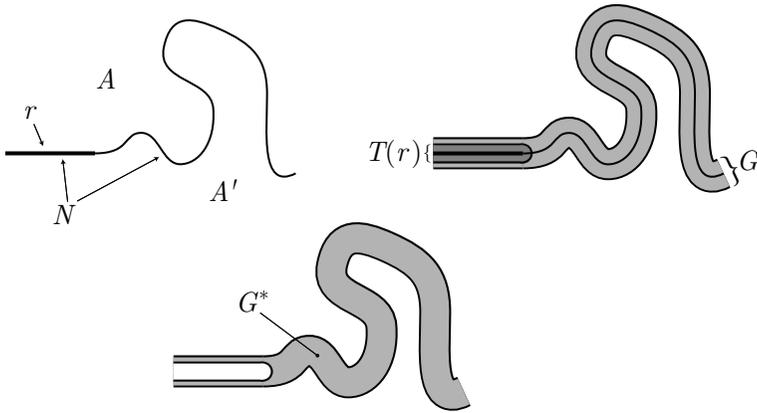


FIGURE 7. Gaskets  $G$ ,  $G^*$  and flange  $T(r)$  for a hyperplane unknotting problem.

*Proof of the HLT (Theorem 6.1) for  $m \geq 3$  and CAT=PL or DIFF.* It suffices to prove that  $A$  and  $A'$ , the closures of the two components of  $\mathbf{R}^m - N$  in  $\mathbf{R}^m$ , are CAT isomorphic to the closed upper half space  $\mathbf{R}_+^m \subset \mathbf{R}^m$ . From  $A$  construct a CSI pair  $\alpha = (A \cup P, P)$  where  $P$  is an open collar neighborhood in  $A'$  of  $\partial A' = N = \partial A$ , the orientation of  $P$  being inherited from  $\mathbf{R}^m$ . Similarly, construct  $\alpha' = (A' \cup P', P')$ .

**Assertion 6.5.** *The CSI composition  $\alpha\alpha'$  of  $\alpha$  and  $\alpha'$  is CAT isomorphic to the trivial CSI pair  $\varepsilon = (\mathbf{R}^m, \mathbf{R}_+^m)$  that is the identity for the CSI operation.*

*Proof of Assertion 6.5.* The key idea is to perceive, embedded in  $\mathbf{R}^m$ , the coarse and fine gaskets for the CSI operation  $\alpha\alpha'$  as suggested by Figure 7. Its coarse gasket can clearly be a bicollar neighborhood  $G$  of  $N$  in  $\mathbf{R}^m$ . We shall prove that a fine gasket is

$$G^* = G - \text{Int } T(r)$$

where  $T(r)$  is a regular neighborhood of  $r$  in  $G$ .

Since  $r$  is, by hypothesis, unknotted in  $N$ , this  $G^*$  is easily seen to be a gasket; it has three boundary components. Note that the three closed complementary components of  $\text{Int } G^*$  in  $\mathbf{R}^m$  are respectively isomorphic to  $A$ ,  $A'$  and  $T(r)$ .

Since  $T(r)$  is CAT degree +1 isomorphic to  $\mathbf{R}_+^m$ , it is a CSI flange, and we conclude from the definition of CSI that the CSI pair  $(\mathbf{R}^m, T(r))$  is (up to CSI pair isomorphism) a CSI product  $\alpha\alpha'$ , whose coarse and fine gaskets are  $G$  and  $G^*$ .

Since  $r$  is, by hypothesis, unknotted in  $\mathbf{R}^m$ , it follows, by PL and DIFF ambient regular neighborhood uniqueness (see Section 3), that the complement of  $\text{Int } T(r)$  in  $\mathbf{R}^m$  is CAT isomorphic to  $\mathbf{R}_+^m$ . Therefore,

$$(\dagger) \quad (\mathbf{R}^m, T(r)) \cong (\mathbf{R}^m, \mathbf{R}_+^m) = \varepsilon,$$

where  $\cong$  denotes CSI pair isomorphism.

Taken together, the last two paragraphs prove the assertion that  $\alpha\alpha' \cong \varepsilon$ .  $\square$

The assertion quickly implies the theorem using the Eilenberg-Mazur swindle. First,  $\varepsilon \cong \alpha\alpha' \cong \alpha'\alpha$  using commutativity, so  $\alpha$  and  $\alpha'$  are mutually inverse. Whence, the infinite product swindle using associativity

$$\alpha \cong \alpha\varepsilon\varepsilon\varepsilon \cdots \cong \alpha(\alpha'\alpha)(\alpha'\alpha) \cdots \cong (\alpha\alpha')(\alpha\alpha')(\alpha\alpha') \cdots \cong \varepsilon\varepsilon\varepsilon \cdots \cong \varepsilon.$$

Also,  $\alpha' \cong \alpha'\varepsilon \cong \alpha'\alpha \cong \varepsilon$ . Thus,  $A$  and  $A'$  are CAT isomorphic to  $\mathbf{R}_+^m$  as required. This establishes the HLT (Theorem 6.1) for  $m \geq 3$  and CAT=PL or CAT=DIFF.  $\square$

*Proof of the HLT (Theorem 6.1) for  $m \geq 3$  and CAT=TOP.* Like Cantrell, we will only use elementary arguments. In particular, recall that, by using our refined version of the definition of CSI for TOP pairs given at the end of Section 2, we have avoided use of the stable homeomorphism theorem (SHT) in establishing the basic properties of CSI.

We now proceed to adapt to TOP the above proof of the DIFF version. It adapts routinely except for the two short paragraphs that apply, to the ray  $r$  in  $\mathbf{R}^m$ , the uniqueness of DIFF regular neighborhoods to deduce the DIFF CSI pair isomorphism  $(\dagger)$ . For TOP, we now establish  $(\dagger)$  using the TOP *open* regular neighborhood uniqueness of Section 3.

We can and do choose a linear structure on  $N$  such that  $r$  is a linear ray in  $N$ . The TOP bicollar neighborhood  $G$  of  $N$  was first

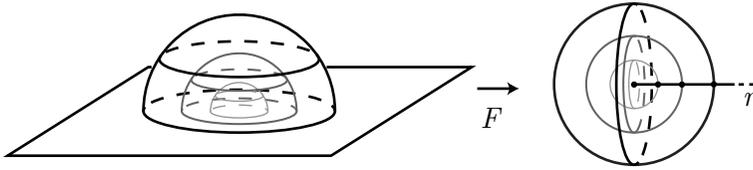


FIGURE 8. Quotient map  $F : \mathbf{R}^{m-1} \times [0, \infty) \rightarrow \mathbf{R}^m$ .

established by Brown in [4]; a pleasant alternative construction is due to R. Connelly, see [36, Essay I, page 40]. This  $G$  can then be viewed as a DIFF gasket of which  $r$  and  $N$  are smooth submanifolds. However, the inclusion of  $G$  into  $\mathbf{R}^m$  is in general not a DIFF embedding. Let  $T(r)$  be a DIFF regular neighborhood of  $r$  in  $G$ , and let  $G^* = G - \text{Int } T(r)$  be the resulting fine gasket.

**Assertion 6.6.** *The closed complement of  $\text{Int } T(r)$  in  $\mathbf{R}^m$  is TOP isomorphic to  $\mathbf{R}_+^m$ . Hence,  $(\dagger)$  holds for TOP CSI pair isomorphism.*

*Proof of Assertion 6.6.* By hypothesis,  $r$  is unknotted in  $\mathbf{R}^m$ . Thus, we can now observe that:

- (1)  $\mathbf{R}^m$  is an open topological mapping cylinder neighborhood of  $r$  in  $\mathbf{R}^m$ .
- (2) The DIFF regular neighborhood  $T(r)$  in  $G$  is a closed mapping cylinder neighborhood of  $r$  in  $\mathbf{R}^m$  with topological frontier  $\partial T(r)$  bicollared in  $\mathbf{R}^m$ .

For (1), define  $f : \mathbf{R}^{m-1} \rightarrow r = [0, \infty)$  by  $f(x) = \|x\|$  which is a proper surjection. The mapping cylinder  $\text{Map}(f)$  embeds homeomorphically onto  $\mathbf{R}^m$  by the quotient map  $F : \mathbf{R}^{m-1} \times [0, \infty) \rightarrow \mathbf{R}^m$  that extends  $(0, x) \mapsto (0, f(x))$  and maps each hemisphere with center the origin onto the full sphere containing it, crushing (only) the hemisphere boundary onto a single point of  $r$  (see Figure 8). Fact (2) follows similarly from our peculiar definition of DIFF regular neighborhood of a ray (see Section 3).

By open mapping cylinder uniqueness (Theorem 3.4), these two facts imply that the closed complement in  $\mathbf{R}^m$  of  $\text{Int } T(r)$  is TOP isomorphic to  $\partial T(r) \times [1, \infty)$ . This completes the proof of the assertion.  $\square$

The proof of the HLT (Theorem 6.1) for TOP now concludes as in the DIFF case.  $\square$

*Remark 6.7.* In the above elementary proof of the TOP version of the HLT (Theorem 6.1), it is *not* proved that the self homeomorphism  $g$  of  $\mathbf{R}^m$  sending  $N$  to a linear hyperplane can be chosen ambient isotopic to a linear map. It is always ambient isotopic; but, to prove this, one needs the SHT of [17, 34].

We close this section with some historical remarks on the Cantrell-Stallings theorem.

(1) Progress towards the TOP theorem from Mazur [39] 1959 to Cantrell's full TOP unknotting theorem in [9] 1963 was incremental. In 1960, Morse [49] extended [39] to prove the TOP version under the extra hypothesis that  $N \cup \infty$  is a TOP bicollared  $(m-1)$ -sphere in the  $m$ -sphere  $\mathbf{R}^m \cup \infty$ . Brown's parallel but amazingly novel article [3] 1960 achieved this, too. Then Brown [4] 1962 proved a collaring theorem that replaced the above bicollaring hypothesis by local flatness in the  $m$ -sphere  $\mathbf{R}^m \cup \infty$ . From 1962 onwards, Cantrell's goal (already reached in 1963) has been viewed as the problem of proving that a codimension 1 sphere in a sphere of dimension  $> 3$  cannot have a single 'singular' point where local flatness fails.

(2) Huebsch and Morse [32] 1962 established the DIFF version under the much stronger unknotting hypothesis that  $N$  be linear outside a bounded set in  $\mathbf{R}^m$ .

(3) Our proof (for any CAT) can be viewed as a radical reorganization using CSI of Cantrell's proof for TOP [9]. On the other hand, it was Stallings [60] who first pointed out the DIFF version, and formulated a version valid in *all* dimensions. The proof of the TOP version requires extra precautions (for us, DIFF gaskets) and extra argumentation (for us, open mapping cylinder neighborhood uniqueness), but, in compensation, it clearly reproves, *ab initio*, the Schoenflies theorem of Mazur [39] and Brown [3, 4].

(4) The apparent novelty, which made us write down the above proof, was our reformulation (circa 2002) of much of the geometry of Cantrell's proof as standard facts about CSI. This explicit use of some sort of connected sum was, of course, suggested by Mazur's pioneering article [39]; compare the 'almost PL' version of the Schoenflies theorem in [51].

(5) CSI itself was not a novelty. Gompf [21] had shown that an infinite CSI of smooth 4-manifolds, each homeomorphic to  $\mathbf{R}^4$ , is well defined. He achieved this by proving a multiple ray unknotting result using finger moves; his proof readily extends to all dimensions  $\geq 4$  (in fact, it is simpler in dimensions  $> 4$ ). Gompf used this observation and the infinite product swindle to show that an exotic  $\mathbf{R}^4$  cannot have an inverse under CSI. The reader can now check this as an exercise.

(6) Stallings [60] deals explicitly only with the DIFF case. He avoids all connected sum notions. Indeed, the basic entity for which he defines an infinite product operation is a (proper) DIFF embedding  $f : \mathbf{R}^{m-1} \rightarrow \mathbf{R}^m$  (with an unknotted ray and  $m \geq 3$ ). Stallings' exposition seems to invite formalization in terms of a *pairwise* CSI operation.

(7) Johannes de Groot in 1972 [25] announced a proof of Cantrell's TOP HLT by generalization of Brown's proof of the TOP Schoenflies theorem. Regrettably, de Groot died shortly thereafter and no manuscript has surfaced since.

**7. Basic ray unknotting in high dimensions  $> 3$ .** The first goal of this section is to explain the well-known fact, mentioned in Remark 6.2 above, that rays in  $\mathbf{R}^m$  are related by an ambient isotopy provided that  $m > 3$ . Then we go on, still assuming  $m > 3$ , to classify so called multirays in terms of the proper homotopy classes of their component rays.

Throughout this section, CAT is one of TOP, PL or DIFF. The following basic result will be needed for 1-manifolds mapping into manifolds of dimension  $m > 3$ .

**Theorem 7.1** (Stable range embedding theorem). *Let  $f : N^n \rightarrow M^m$  be a proper continuous map of CAT manifolds, possibly with boundary. If  $2n + 1 \leq m$ , then  $f$  is properly homotopic to a CAT embedding  $g : N \rightarrow M$  such that  $g(N)$  lies in  $\text{Int } M$ . Further, if  $2n + 2 \leq m$  and  $g'$  is a second such embedding properly homotopic to  $f$ , then a CAT ambient isotopy  $h_t : M \rightarrow M$ ,  $0 \leq t \leq 1$ , exists such that  $h_0 = \text{id}|_M$  and  $h_1 g = g'$ .*

For CAT=PL or CAT=DIFF, the proof is a basic general position argument that can be found in many textbooks. Early references are [2, 65].

For TOP, the proof is still surprisingly difficult. One needs a famous method of Homma from 1962 [30], as applied by Gluck [19, 20]. Many expositions of these types of results (in particular [20]) are given in a compact relative form, from which one has to deduce the stated noncompact, nonrelative but proper version by a classical argument involving a skeletal induction in the nerve of a suitable covering (see [36, Essay I, Appendix C]).

Next, we show that, in some cases of current interest, all rays are properly homotopic.

**Lemma 7.2** (Simplest proper ray homotopies). *Let  $X$  be locally arcwise connected and locally compact. Suppose  $X$  admits a connected closed collar neighborhood  $Y \times [0, \infty)$  of Alexandroff infinity. Then any two proper maps  $[0, \infty) \rightarrow X$  are properly homotopic.*

*Proof.* Any proper map  $f : [0, \infty) \rightarrow X$  is properly homotopic to one with an image in the closed subset  $Y \times [0, \infty) \subset X$ , so we can and do assume that  $X$  is  $Y \times [0, \infty)$ .

Then, writing  $f(0) = (y, t_0) \in Y \times [0, \infty)$ , it is easy to construct an explicit proper homotopy of  $f$  to the proper continuous radial embedding  $r_y : [0, \infty) \hookrightarrow X = Y \times [0, \infty)$  sending  $t \mapsto (y, t)$  for all  $t$ .

Finally, for any two points  $y$  and  $y'$  in  $Y$ , there is a path from  $y$  to  $y'$  in  $Y$ , and any such path provides an explicit proper homotopy from  $r_y$  to the similarly defined radial embedding  $r_{y'}$ .  $\square$

These last two results, when combined with the Cantrell-Stallings theorem as stated in the last section (Theorem 6.1), yield the following hyperplane linearization theorem already announced there.

**Theorem 7.3.** *For  $m \neq 3$ , any CAT submanifold  $N$  of  $\mathbf{R}^m$  that is isomorphic to  $\mathbf{R}^{m-1}$  is unknotted in the sense that there is a CAT automorphism  $h$  of  $\mathbf{R}^m$  such that  $h(N) = \mathbf{R}^{m-1} \times 0 \subset \mathbf{R}^m$ .*

*Remarks 7.4.* (1) Remember that, by convention, a CAT submanifold is a closed subset and is assumed CAT locally flat unless the contrary is explicitly stated.

(2) The case CAT=TOP of Theorem 7.3 is Cantrell's result as he formulated it. Beware that (still today) any completely bootstrapping proof seems to require an exposition of Homma's method.

(3) It is well known that a proper ray (any CAT) may be knotted in  $\mathbf{R}^3$ . Fox and Artin [16, Example 1.2] exhibited the first such ray, Alford and Ball [1] produced infinitely many knot types and conjectured uncountably many exist, and McPherson [41] published a proof of this conjecture (earlier, Giffen, 1963, Sikkema, Kinoshita, and Lomonaco, 1967, and McPherson, 1969, had announced proofs [5, page 273]). The boundary of a closed regular neighborhood of any such knotted ray is a knotted hyperplane in  $\mathbf{R}^3$ . Still, even in this dimension, the knot type of any CAT hyperplane  $N \subset \mathbf{R}^3$  is determined by the knot type in  $\mathbf{R}^3$  of any CAT ray  $r \subset N$  [58] (see also [26]); in fact,  $N$  is ambient isotopic to the boundary of a CAT closed regular neighborhood of  $r$  in  $\mathbf{R}^3$  [8]. Thus, one of the two closed complementary components of  $N$  in  $\mathbf{R}^3$  is CAT isomorphic to  $\mathbf{R}_+^3$ .

(4) Here is an immediate corollary for  $\text{CAT} = \text{DIFF}$  that concerns the still mysterious dimension 4. Suppose that  $N^3 \subset S^4$  is a smoothly embedded 3-sphere such that the pair  $(S^4, N^3)$  is *not* DIFF isomorphic to  $(S^4, S^3)$  and thus is a counterexample to the unsettled DIFF 4-dimensional Schoenflies conjecture. Then, nevertheless, for any point  $p$  in  $N^3$ , one has  $(S^4 - p, N^3 - p) \cong (\mathbf{R}^4, \mathbf{R}^3)$ .

(5) We have seen that the Cantrell-Stallings unknotting theorem is closely related to the fact that: *if  $\alpha := (M, P)$  is a dimension  $m$  CAT CSI pair that has an inverse up to degree  $+1$  isomorphism in the commutative semigroup of isomorphism classes of CAT CSI pairs of dimension  $m \geq 3$  under CSI sum, then  $(M, P)$  is in the identity class, namely, that of  $(\mathbf{R}^m, \mathbf{R}_+^m)$ .* Thus, it is perhaps of interest to ask about other algebraically expressible facts about this semigroup. For example: *is it true that  $\alpha \cong \alpha\beta$  always implies that  $\alpha \cong \alpha\beta^\infty$ ?* Curiously, this is false for certain  $(M, P)$  where  $M$  has more than one end, as Figure 9 indicates.

Although this figure is for dimension 2, it clearly has analogs in all dimensions  $> 2$ . Is this implication true at least when  $M$  has one end? Or when  $M$  is the interior of a compact manifold?

This concludes our exposition of the Cantrell-Stallings theorem.

**8. Singular and multiple rays.** This section shows that multiple rays embed and unknot much like single rays. We define a *singular ray* in a locally compact space  $X$  to be a proper continuous map

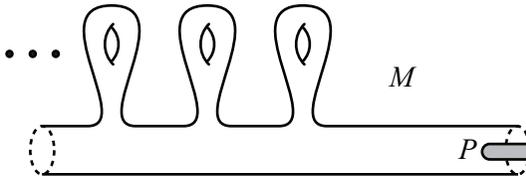


FIGURE 9. CSI pair  $(M, P)$  where  $M$  has two ends and one end is collared.

$[0, \infty) \rightarrow X$ . In Section 9, singular rays will be a tool for unknotting multiple hyperplanes in dimensions  $> 3$ .

**Lemma 8.1.** *Let  $f_i : [0, \infty) \rightarrow X$ , with  $i$  varying in the finite or countably infinite discrete index set  $S$ , be singular rays in a locally compact, sigma compact space  $X$ . Then, for each  $i \in S$ , one can choose a proper homotopy of  $f_i$  to a singular ray  $f'_i$  such that the rule  $(i, x) \mapsto f'_i(x)$  defines a proper map  $f' : S \times [0, \infty) \rightarrow X$ .*

*Proof.* The choice  $f_i = f'_i$  will do, in case  $S$  is finite. When  $S$  is infinite, we can assume  $S = \mathbf{Z}_+$ . Then, choose in  $X$  a sequence of compacta  $\emptyset = K_1 \Subset K_2 \Subset K_3 \Subset \dots$  with  $X = \cup_j K_j$ . By properness of  $f_i$ ,  $a_i$  in  $[0, \infty)$  exists so large that  $f_i([a_i, \infty)) \subset X - K_i$ . Define  $f'_i$  to be  $f_i$  precomposed with the retraction  $[0, \infty) \rightarrow [a_i, \infty)$ .

It is easily seen that  $f'_i$  is properly homotopic to  $f_i$ . The properness of the resulting  $f'_i$  now follows. Indeed, if  $K \subset X$  is compact, then  $K$  lies in the interior of  $K_i$  for some  $i$ ; hence,  $f'_j([0, \infty)) \cap K = \emptyset$  for  $j > i$ . Thus, the preimage  $f'^{-1}(K)$  in  $S \times [0, \infty)$  meets  $j \times [0, \infty)$  only for  $j \leq i$ . But, the intersection  $f'^{-1}(K) \cap \{1, 2, \dots, i\} \times [0, \infty)$  is compact by the finite case.  $\square$

Here is a key lemma concerning just one singular ray that will help to deal with infinitely many rays.

**Lemma 8.2.** *Let  $K$  be a given compact subset of a locally compact, sigma compact space  $X$ , and let  $f$  and  $f'$  be singular rays in  $X$  whose images are disjoint from  $K$ . If  $f$  and  $f'$  are properly homotopic in  $X$ , then the proper homotopy can be (re)chosen to have image disjoint from  $K$ .*

*Proof.* If  $h_t : [0, \infty) \rightarrow X$ ,  $0 \leq t \leq 1$ , is a proper homotopy from  $f = h_0$  to  $f' = h_1$ , then its properness assures that, for some  $d \geq 0$ ,

the image  $h_t([d, \infty))$  is disjoint from  $K$  for all  $t$ . But, the singular ray  $f$  is proper homotopic in the complement of  $K$  to the singular ray  $\widehat{f}$ , that is,  $f|_{[0, \infty)}$  precomposed with the retraction  $[0, \infty) \rightarrow [d, \infty)$ . This is similar for  $f'$ . Shunting together these three proper homotopies, one obtains the asserted proper homotopy.  $\square$

**Lemma 8.3.** *Let  $f_i$  and  $f'_i$ , with  $i$  varying in the finite or countably infinite discrete set  $S$ , be two indexed sets of singular rays in the connected, locally compact, sigma compact space  $X$ . Suppose that the two continuous maps  $f$  and  $f'$  from  $S \times [0, \infty)$  to  $X$  defined by the rules  $(i, x) \mapsto f_i(x)$  and  $(i, x) \mapsto f'_i(x)$  are both proper. Suppose also that  $f_i$  is proper homotopic to  $f'_i$  for all  $i \in S$ . Then, a proper homotopy  $h_t : S \times [0, \infty) \rightarrow X$ ,  $0 \leq t \leq 1$ , exists that deforms  $h_0 = f$  to  $h_1 = f'$ .*

*Proof.* We propose to define the needed proper homotopy  $h_t$  by choosing, for  $i \in S$ , suitable proper homotopies  $h_{i,t}$  from  $f_i$  to  $f'_i$  and then defining  $h_t$  by setting  $h_t(i, x) = h_{i,t}(x)$  for all  $i \in S$ , all  $t \in [0, 1]$ , and all  $x \in [0, \infty)$ . The choices aim to ensure that  $h_t$  is a proper homotopy—which means that the rule  $(t, i, x) \mapsto h_t(i, x)$  is proper as a map  $[0, 1] \times S \times [0, \infty) \rightarrow X$ .

If  $S$  is finite, any choices will do. But, if  $S$  is infinite, then bad choices abound. For example,  $h_t$  is not proper if every homotopy  $h_{i,t}(x)$  meets a certain compactum  $K$ .

If  $S$  is infinite, we now specify choices that do the trick. Without loss of generality, assume  $S = \mathbf{Z}_+$ . Let  $\emptyset = K_1 \Subset K_2 \Subset K_3 \Subset \dots$  be an infinite sequence of compacta with  $X = \cup_j K_j$ . For each  $i \in S$ , let  $J(i)$  be the greatest positive integer such that the images of the singular rays  $f_i$  and  $f'_i$  are both disjoint from  $K_{J(i)}$ . Since  $f$  and  $f'$  are proper,  $J(i)$  tends to infinity as  $i$  tends to infinity. Use Lemma 8.2 to choose the proper homotopy  $h_{i,t}$  from  $f_i$  to  $f'_i$  to have image disjoint from  $K_{J(i)}$ . Then, the properness of the resulting  $h_t$  is verified as in the proof of Lemma 8.1.  $\square$

*Remark 8.4.* Lemmas 8.1 to 8.3 above hold good with  $[0, \infty)$  replaced by its product with (varying) compacta.

Define a *multiray* in the CAT manifold  $M^m$  to be a CAT submanifold lying in  $\text{Int } M$ , each component of which is a ray. Combining the stable range embedding theorem (Theorem 7.1) with Lemmas 8.1–8.3 concerning proper maps, we get:

**Proposition 8.5.** (Classifying multirays via proper homotopy). *Let  $M^m$  be a connected noncompact CAT manifold, and let  $f_i$  be singular rays where  $i$  ranges over a finite or countably infinite index set  $S$ . If  $m \geq 3$ , then  $f_i$  is properly homotopic to a CAT embedding  $g_i$  onto a ray, such that the rules  $(i, x) \mapsto g_i(x)$  collectively define a (proper) CAT embedding  $g : S \times [0, \infty) \rightarrow M$  with the image a multiray. Furthermore, if  $m > 3$  and  $g'_i$  is an alternative choice of the ray embeddings  $g_i$ , resulting in the alternative CAT embedding  $g'$  onto a multiray, then an ambient isotopy  $h_t : M \rightarrow M$ ,  $0 \leq t \leq 1$ , exists such that  $h_0 = \text{id}|_M$  and  $h_1 g = g'$ .  $\square$*

**9. Multiple component hyperplane embeddings.** In this section we investigate proper CAT embeddings into  $\mathbf{R}^m$  of a disjoint sum of at most countably many disjoint hyperplanes, each isomorphic to  $\mathbf{R}^{m-1}$ . Indeed, every closed subset of a separable metric space is separable.

For CAT=TOP we will, for the first time, make essential use of the stable homeomorphism theorem (SHT) to show that every self homeomorphism of  $\mathbf{R}^k$  is ambient isotopic to a linear one [17, 34]; this is equivalent to  $\pi_0(\text{STop}(k)) = 0$ , where  $\text{STop}(k)$  is the group of orientation preserving self homeomorphisms of  $\mathbf{R}^k$  endowed with the compact open topology. Not to do so would lead to pointless hairsplitting.

In these circumstances, we can and do revert to unrefined versions of the definition for TOP of the CSI operation and its related constructions. We use the following lemma.

**Lemma 9.1.** *If  $G$  and  $G'$  are TOP gaskets and  $f : \partial G \rightarrow \partial G'$  is a degree +1 TOP isomorphism of their boundaries, then  $f$  extends to a degree +1 TOP isomorphism  $F : G \rightarrow G'$ .*

*Proof.* By definition of gasket (see Section 2), we may assume  $G$  and  $G'$  are linear gaskets. By the SHT, we can isotop  $f$  to a DIFF isomorphism  $f'$ . This  $f'$  extends to a degree +1 DIFF isomorphism  $F' : G \rightarrow G'$  by the DIFF version of this lemma (Corollary 4.5 above). Using closed collars of  $\partial G$  and  $\partial G'$ , we easily construct the asserted TOP isomorphism  $G \rightarrow G'$ .  $\square$

A *multiple hyperplane* is a properly embedded submanifold  $N$  of  $\mathbf{R}^m$  where  $N$  is the disjoint union of components  $N_i \cong \mathbf{R}^{m-1}$  for  $i \in S$ ,

and  $S$  is a nonempty countable index set. We say that  $G$ , the closure of a component of  $\mathbf{R}^m - N$  in  $\mathbf{R}^m$ , is *docile* if it is a gasket, and we say that  $N$  itself is *docile* if the closure of every such component is docile.

Given any multiple hyperplane  $N$  in  $\mathbf{R}^m$ , we can construct a canonical simplicial tree  $T$  as follows. The vertices  $V$  of  $T$  are the closures of the complementary components of  $N$  in  $\mathbf{R}^m$ . An edge is a component  $N_i$  of  $N$ , and it joins the two vertices  $u, v \in V$  whose intersection is  $N_i$ . The tree  $T$  is clearly well-defined by the pair  $(\mathbf{R}^m, N)$  up to tree isomorphism; it is the nerve of the covering of  $\mathbf{R}^m$  by the closures of the components of  $\mathbf{R}^m - N$ . Also,  $T$  is at most countable, but it is not necessarily locally finite. If  $m = 2$ , then these trees are naturally planar as the edges at each vertex are cyclically ordered.

Conversely, given such a tree  $T$  (planar in case  $m = 2$ ), there is a natural recipe to construct a multiple hyperplane  $N$  in  $\mathbf{R}^m$  where the closure of each complementary component is a gasket as follows. For each vertex  $v_k \in V$ , pick a gasket  $G_k$  with boundaries corresponding bijectively to the edges incident with  $v_k$  in  $T$ . Gluing these gaskets together according to  $T$  gives a composite gasket  $TG$  with empty boundary.

It was established in proving the associativity property of CSI that there is a CAT manifold isomorphism  $TG \rightarrow \mathbf{H}^m$  sending each vertex gasket in  $TG$  to a linear gasket in  $\mathbf{H}^m$  and, hence, each edge hyperplane to a hyperbolic hyperplane in  $\mathbf{H}^m$  (see Lemma 5.2 above). Further, such an isomorphism is unique up to degree +1 CAT isomorphism of  $\mathbf{H}^m$ .

We now summarize these observations, where CAT is TOP, PL or DIFF.

**Theorem 9.2** (Multiple hyperplane linearization theorem (MHLT)). *For  $m$  distinct from 3, every CAT multiple hyperplane embedding  $N$  in  $\mathbf{R}^m$  is docile. Hence, for  $m > 3$ , such embeddings are naturally classified modulo ambient degree +1 CAT automorphism by arbitrary countable simplicial trees modulo simplicial tree automorphisms. For  $m = 2$  (and only  $m = 2$ ) one must use planar trees and their planar tree automorphisms (where planar here means that, at each vertex, the edges are cyclicly ordered).*

**Corollary 9.3** (Gasket recognition theorem (GRT)). *Consider a CAT  $m$ -manifold with nonempty boundary whose interior is isomorphic to*

$\mathbf{R}^m$ , and for which every boundary component is isomorphic to  $\mathbf{R}^{m-1}$ . Exclude the case  $m = 3$ . Then  $M$  is isomorphic to a linear gasket.  $\square$

*Proof of the GRT* (Corollary 9.3).  $\text{Int } M$  is always isomorphic to the manifold obtained by adding to  $M$  an external open collar along  $\partial M$ .  $\square$

**Corollary 9.4.** *With the same data as in the MHLT and assuming  $m \geq 4$ , the pair  $(\mathbf{R}^m, N)$  is CAT isomorphic to a Cartesian product*

$$(\mathbf{H}^2, N') \times \mathbf{R}^{m-2}$$

where each component of  $N'$  is a hyperbolic line.  $\square$

*Proof of the MHLT* (Theorem 9.2) for  $m > 3$  and  $\text{CAT}=\text{PL}$  or  $\text{DIFF}$ . Let  $G$  be the closure of a component of  $\mathbf{R}^m - N$  in  $\mathbf{R}^m$ . Reindex so that  $N_i, i \in S$ , are the boundary components of  $G$ . For each  $N_i$ , let  $V_i$  denote the closed component of  $\mathbf{R}^m - \text{Int } G$  with boundary  $N_i$ . Each  $N_i$  is unknotted in  $\mathbf{R}^m$  by the CAT HLT (Theorem 7.3). Therefore, for each  $i \in S$  there is a CAT proper ray  $r_i \subset \text{Int } V_i$  so that  $V_i$  is a CAT regular neighborhood of  $r_i$  in  $\mathbf{R}^m$ . As  $N \subset \mathbf{R}^m$  is a proper submanifold, the union of the rays  $r_i$  is a proper multiray in  $\mathbf{R}^m$ .

Choose  $G' \subset \mathbf{H}^m$  a linear gasket with boundary hyperplanes  $N'_i, i \in S$ . For each  $N'_i$ , let  $V'_i$  denote the closed component of  $\mathbf{H}^m - \text{Int } G'$  with boundary  $N'_i$ , and let  $r'_i \subset \text{Int } V'_i$  be a radial ray. Plainly,  $V'_i$  is a CAT regular neighborhood of  $r'_i$  for each  $i \in S$ , and the union of the rays  $r'_i$  is a proper multiray in  $\mathbf{H}^m$ .

Choose a CAT isomorphism  $\psi : \mathbf{R}^m \rightarrow \mathbf{H}^m$ . CAT proper multirays unknot in  $\mathbf{H}^m, m > 3$ , by the basic CAT stable range embedding theorem (Theorem 7.1), proved by general position, and Lemmas 7.2 and 8.3. Thus, there is an ambient isotopy of  $\mathbf{H}^m$  carrying  $\psi(r_i)$  to  $r'_i$  for all  $i \in S$  simultaneously. So, we may as well assume  $\psi(r_i) = r'_i$  for  $i \in S$ . By PL and DIFF regular neighborhood ambient uniqueness (see Section 3), we may further assume that  $\psi(V_i) = V'_i$  for all  $i \in S$ . Then,  $\psi|_G : G \rightarrow G'$  is a CAT isomorphism as desired.  $\square$

*Proof of the MHLT* (Theorem 9.2) for  $m > 3$  and  $\text{CAT}=\text{TOP}$ . Again, let  $G$  be the closure of a component of  $\mathbf{R}^m - N$  in  $\mathbf{R}^m$  and reindex so that  $N_i, i \in S$ , are the boundary components of  $G$ . We have three cases depending upon whether the number  $|S|$  of boundary components of  $G$  is 1, 2, or  $> 2$ .

Case  $|S| = 1$ . This is exactly Cantrell's TOP HLT (Theorem 7.3).

Case  $|S| = 2$ . This case is well known as the *Slab Theorem* and is a worthy sequel by Greathouse [24], 1964, to Cantrell's TOP HLT (Theorem 7.3), so we include a proof. Greathouse deduced it from results then recently established, together with the following (for  $m > 3$ ), then unproved.

**Theorem 9.5** (Annulus conjecture (AC( $m$ ))). *If  $S_1$  and  $S_2$  are two disjoint locally flatly embedded  $(m - 1)$ -spheres in  $S^m$ , and  $X$  is the closure of the component of  $S^m - (S_1 \cup S_2)$  with  $\partial X = S_1 \sqcup S_2$ , then  $X$  is homeomorphic to the standard annulus  $S^{m-1} \times [0, 1]$ .*

This annulus conjecture was later proved, along with the SHT, in [34] 1969 for  $m > 4$ , and in [17], 1990, for  $m = 4$  (see also [15]). The already proved results used in [24] included Cantrell's TOP HLT, that we have reproved (Theorem 7.3), and the following, proved by Cantrell and Edwards [12], 1963.

**Lemma 9.6** (Arc flattening lemma). *If a compact arc  $A$  topologically embedded in  $S^m$ ,  $m > 3$ , is locally flat except possibly at one interior point  $P$ , then  $A$  is locally flat also at  $P$ .*

Assuming these tools for the moment, we now give:

*Proof of the slab theorem.* We consider the sphere  $S^m$  to be  $\mathbf{R}^m \cup \infty$ . Let  $G_i$ ,  $i = 1, 2$ , be the components of  $\mathbf{R}^m - \text{Int } G$ . By the TOP HLT (Theorem 7.3), each  $G_i \cong \mathbf{R}_+^m$ . Hats will indicate the adjunction of the point  $\infty \in S^m$ . Enlarge  $\widehat{G} := G \cup \infty$  by adding to it a closed collar  $C_i$  of the  $(m - 1)$ -sphere  $\partial \widehat{G}_i$  in  $\widehat{G}_i$ , for  $i = 1, 2$ . Denote the result  $X := \widehat{G} \cup C_1 \cup C_2$ . This is a TOP submanifold of  $S^m$  with boundary two  $(m - 1)$ -spheres  $S_1$  and  $S_2$ , where  $S_i$ , for  $i = 1$  and  $2$ , is the component of  $\partial C_i$  disjoint from  $\widehat{G}$  (see Figure 10).

The theorem AC( $m$ ) (Theorem 9.5) tells us that  $X \cong S^{m-1} \times [0, 1]$ . Furthermore, we have collaring identifications  $C_i = S_i \times [0, 1]$ . Consider the locally flat arc  $A_i$  that is the arc fiber of the collaring  $C_i = S_i \times [0, 1]$  that contains  $\infty \in S^m$ . Clearly,  $A_1 \cap A_2 = \infty$ ; thus  $A := A_1 \cup A_2$  is an arc in  $X$  that is locally flat except possibly at  $\infty \in S^m$ . By the above arc flattening lemma (Lemma 9.6),  $A$  is locally flat at  $\infty$ ; hence, it is a locally flat 1-submanifold of  $X$  joining the two boundary components

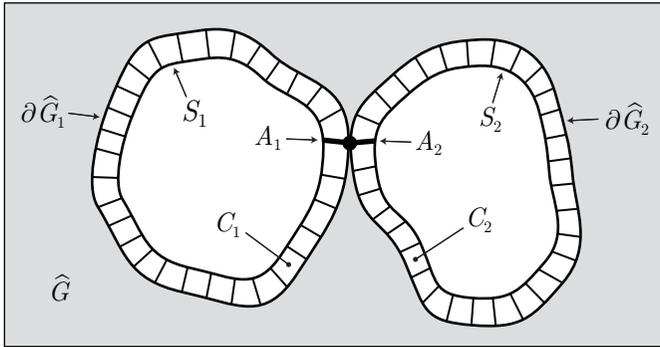


FIGURE 10. Almost global view of  $\widehat{G}$  in  $S^m = \mathbf{R}^m \cup \infty$ , focused on the point  $\infty = A_1 \cap A_2$ .

of  $X$ . Note that  $G \cong X - A$  by Brown’s collaring uniqueness theorem [4].

By the uniqueness clause of the elementary (but subtle!) TOP version of the stable range embedding theorem (Theorem 7.1), any two such arcs are related by a TOP automorphism of  $X \cong S^{m-1} \times [0, 1]$ . Thus, the complement  $X - A$  is homeomorphic to  $\mathbf{R}^{m-1} \times [0, 1]$ .  $\square$

*Proof of the arc flattening lemma.* Split  $A$  at  $P$  to get two compact arcs  $A_1$  and  $A_2$  with  $A_1 \cap A_2 = P$ .

**Assertion 9.7.** *A compact locally flat  $n$ -ball neighborhood  $B$  of  $\text{Int } A_1$  exists such that  $A_1$  is unknotted in  $B$  and  $B$  is disjoint from  $\text{Int } A_2$ .*

*Proof of Assertion 9.7.* In our one application of the arc flattening lemma above (namely to prove the Slab Theorem),  $B$  can obviously be any tubular neighborhood of  $A_1$  in  $C_1$  derived from the product structure  $C_1 = S_1 \times [0, 1]$ . Thus, we leave the full proof of this assertion to the interested reader with just this hint:  $B$  can in general be the closure in  $S^m$  of a suitably tapered trivial normal tubular neighborhood of  $\text{Int } A_1$  in  $S^m$  (see [38]).  $\square$

Now, by the TOP Schoenflies theorem,  $S^m - \text{Int } B$  is also an  $m$ -ball  $B'$  in  $S^m$ . In  $B'$  the second arc  $A_2$  is embedded in a manner that is locally flat except possibly at  $P \in \partial B'$ . To the non-compact TOP manifold

$B' - P \cong \mathbf{R}_+^m$ , we apply the uniqueness clause of the stable range embedding theorem (Theorem 7.1); we conclude, on recompactifying in  $S^m$ , that the arc  $A_2$  is unknotted in  $B'$ . It follows that the arc  $A := A_1 \cup A_2$  is locally flat in  $S^m$ . This completes our proof of the arc flattening lemma (Lemma 9.6).  $\square$

Assuming AC( $m$ ) (now known!), this completes the proof of the slab theorem which is the MHLT (Theorem 9.2) for the case when  $G$  has two boundary components.  $\square$

*Remark 9.8.* Greathouse [23], 1964, also proved that the slab theorem in dimension  $m$  implies AC( $m$ ), granting results known in 1964 that we have mentioned. Hints: given an  $m$ -annulus  $X$  in  $S^m$ , form a locally flat arc  $A \subset X$  joining the two boundary  $(m-1)$ -spheres. Show that  $A$  is *cellular* (i.e., an intersection of compact  $m$ -cell neighborhoods in  $S^m$ ) so that the quotient space  $(S^m/A)$  is homeomorphic to  $S^m$ , and apply the slab theorem to show that  $X - A \cong \mathbf{R}^{m-1} \times [0, 1]$ . Deduce that  $X \cong S^{m-1} \times [0, 1]$  with the help of collarings and the Mazur-Brown Schoenflies theorem.

*Remark 9.9.* An easy argument shows that AC( $n$ ) (Theorem 9.5),  $n = 1, \dots, m$ , together imply the following.

**Theorem 9.10** (Stable homeomorphism conjecture (SHC( $m$ ))). *For any homeomorphism  $h : \mathbf{R}^m \rightarrow \mathbf{R}^m$ , a homeomorphism  $h' : \mathbf{R}^m \rightarrow \mathbf{R}^m$  exists that coincides with  $h$  near the origin and with the identity map outside a bounded set.*

Hint: For this implication, you will need some Alexander isotopies. Exactly this form of the SHC( $m$ ) was proved for  $m \geq 5$  by Kirby in [34].

*Remark 9.11.* An easy argument establishes the implication SHC( $m$ )  $\Rightarrow$  AC( $m$ ).

*Proof of the MHLT (Theorem 9.2) for CAT=TOP and  $|S| > 2$ .* Let  $G$  be the closure in  $\mathbf{R}^m$  of a component of  $\mathbf{R}^m - N$ . Let  $N_i$ ,  $i \in S$ , be an indexing of the components of  $\partial G$ . For each  $N_i$ , let  $V_i$  denote the closed component of  $\mathbf{R}^m - \text{Int } G$  with boundary  $N_i$ . By the TOP HLT (Theorem 7.3), each  $V_i$  is TOP isomorphic to closed upper half

space  $\mathbf{R}_+^m$ . It is straightforward to produce, for each  $i \in S$ , a DIFF proper ray  $r_i \subset \text{Int } V_i$ . Let  $T(r_i) \subset \text{Int } V_i$  be a DIFF (closed) regular neighborhood of  $r_i$ . The boundary  $H_i$  of  $T(r_i)$  is a DIFF hyperplane. By the slab theorem, the closure of the region between  $H_i$  and  $N_i$  is TOP isomorphic to  $\mathbf{R}^{m-1} \times [0, 1]$ . This isomorphism yields an obvious isotopy of  $N_i$  to  $H_i$  for each  $i \in S$ . Using disjoint collars of the  $H_i$  and  $N_i$ , these isotopies readily extend to an ambient isotopy of  $\mathbf{R}^m$  which carries the collection  $N_i, i \in S$ , to the DIFF collection  $H_i, i \in S$ . The result now follows from the DIFF proof of the MHLT (Theorem 9.2) above.  $\square$

This completes the proof of the MHLT (Theorem 9.2) for  $m > 3$  and  $\text{CAT}=\text{TOP}$ .  $\square$

*Proof of the MHLT (Theorem 9.2) for  $m = 2$ .* We begin with the following.

**Observation 9.12.** *By triangulation and smoothing theorems for dimension 2 that we refer to collectively as the 2-Hauptvermutung (see [48, 55]), it suffices to establish the MHLT (Theorem 9.2) for any one of the three categories  $\text{CAT} = \text{DIFF}, \text{PL},$  or  $\text{TOP}$ .*

We work in the smooth category. The DIFF proof of the MHLT (Theorem 9.2) already given for  $m > 3$  easily adapts to  $m = 2$  using the following.

**Theorem 9.13** (Multiray radialization theorem in  $\mathbf{R}^2$  (2-MRT)). *Let  $L \subset \mathbf{R}^2$  be a DIFF multiray. Then there exists a degree +1 DIFF automorphism  $g$  of  $\mathbf{R}^2$  such that  $g(L)$  is a radial multiray.*

*Proof of the 2-MRT (Theorem 9.13).* Translate so that  $L$  misses the origin. Morse theory tells us that, by a small smooth perturbation of  $L$  in  $\mathbf{R}^2$ , we may assume that

$$\begin{aligned} \mathbf{R}^2 &\xrightarrow{f} \mathbf{R} \\ x &\longmapsto |x| \end{aligned}$$

restricts to a Morse function on  $L$  with distinct critical values, cf. [46].

**Assertion 9.14.** *By an ambient isotopy, we may assume that, on each component  $r$  of  $L$ , an absolute minimum of the restriction  $f|_r$  is attained at the point  $\partial r$  only, and this point is noncritical for  $f|_r$ .*

*Proof of Assertion 9.14.* For each component  $r_i$  of  $L$  for which  $f(\partial r_i)$  is not the unique minimum point  $m_i$  of  $f$  on  $r_i$ , consider a small smooth regular neighborhood  $N_i$  of the interval  $K_i$  in  $r_i$  that joins  $m_i$  to  $\partial r_i$ . These  $N_i$  can be chosen so small that their union  $N$  is a disjoint sum of these  $N_i$ . Then, independent smooth isotopies, each with support in one  $N_i$ , together establish the assertion (cf. [47, pages 22–24]).  $\square$

Seen in a nutshell, the remainder of our proof plan is as follows:

(1) Let  $L_1 = L$  and do the following steps (a) and (b) for  $i = 1, 2, 3, \dots$  until all critical points of  $f|_{L_i}$  are eliminated:

(a) Pick an appropriate local minimum  $u_0$  and maximum  $u_1$  of  $f|_{L_i}$  and find a degree 1 diffeomorphism  $h_i$  of  $\mathbf{R}^2$  so that the critical points of  $f|_{h_i(L_i)}$  are the critical points of  $f|_{L_i}$ , with the exception of  $u_0$  and  $u_1$ .

(b) Let  $L_{i+1} = h_i(L_i)$ .

(2) Using special properties of the  $h_i$ , show that the (probably infinite) composition  $h = \dots h_3 h_2 h_1$  is a diffeomorphism.

(3) Since  $f|_{h(L)}$  has no critical points, conclude that a further diffeomorphism will straighten  $h(L)$ , by integrating a vector field on  $\mathbf{R}^2$  that is tangent to  $h(L)$  and is transverse to the level spheres of  $f$ .

Step (3) follows from:

**Lemma 9.15.** *Let  $M$  be a closed DIFF manifold, and let*

$$p: M \times [0, \infty) \longrightarrow [0, \infty)$$

*be projection. Suppose  $L \subset M \times [0, \infty)$  is a DIFF multiray so that  $p|_L$  has no critical points. If  $L$  intersects  $M \times 0$ , assume further that  $L$  is straight near  $M \times 0$ , i.e.,  $L \cap M \times [0, \epsilon] = F \times [0, \epsilon]$  for some  $\epsilon > 0$  and some finite set  $F \subset M$ . Then there is a DIFF automorphism  $h$  of  $M \times [0, \infty)$  so that:*

(1)  $h(L)$  is a disjoint union of straight rays  $x_i \times [t_i, \infty)$ .

(2)  $ph = p$ .

(3)  $h$  is the identity near  $M \times 0$ .

*Proof.* Let  $v$  be a nowhere 0 tangent vector field on  $L$ . Since  $p|_L$  has no critical points, we know  $v(p)$  is nowhere 0 on  $L$ . After negating  $v$  on some of the components of  $L$ , we may thus assume  $v(p) > 0$  everywhere on  $L$ . Extend  $v$  to a vector field  $v$  on a neighborhood  $U$  of  $L$ . If  $L$  intersects  $M \times 0$ , we may assume  $v = (0, 1)$  near  $M \times 0$ . We may suppose after shrinking  $U$  and replacing  $v$  by  $v/v(p)$  that  $v(p) = 1$  everywhere on  $U$ . Let  $f: M \times [0, \infty) \rightarrow [0, 1]$  be a smooth function with support in  $U$  and equal to 1 on  $L$ . Define a vector field  $w$  on  $M \times [0, \infty)$  by  $w = fv + (1-f)(0, 1)$ . Note that  $w(p) = fv(p) + (1-f) = 1$  everywhere. Let  $\phi((x, s), t)$  be the maximal flow obtained by integrating  $w$ . Since  $w$  is  $(0, 1)$  near  $M \times 0$ , we know  $\phi((x, 0), t) = (x, t)$  for small  $t \geq 0$ . Since  $w(p) = 1$ , we know  $p\phi((x, s), t) = p(x, s) + t = s + t$  everywhere  $\phi$  is defined. As  $M$  has empty boundary, for each  $(x, s) \in M \times (0, \infty)$  an  $\varepsilon > 0$  exists such that  $\phi$  is defined on  $(x, s) \times (-\varepsilon, \varepsilon)$ . For each  $(x, s) \in M \times [0, \infty)$ , the last three sentences and compactness of  $M \times [s, s + 1]$  imply that  $\phi$  is defined on  $(x, s) \times [0, 1]$ . Fitting these solutions together, we see that  $\phi$  is defined on  $(M \times [0, \infty)) \times [0, \infty)$  (cf. [28, pages 149–151]). Since  $w$  is tangent to  $L$ , we know that if  $\phi((x, s), t_0) \in L$  then there is an interval  $[a, \infty)$  so that  $\phi((x, s), t) \in L$  for all  $t \in [a, \infty)$ , and in fact  $\phi((x, s) \times [a, \infty))$  is a connected component of  $L$ . We now define  $h$  by specifying  $h^{-1}(x, t) = \phi((x, 0), t)$  or equivalently,  $h(x, t) = (q\phi((x, t), -t), t)$  where  $q: M \times [0, \infty) \rightarrow M$  is projection.  $\square$

Step (2) will follow from Proposition 9.16 below with  $X := \mathbf{R}^2$  and  $U_j$  the open disc of radius  $j$ . To ensure applicability of this proposition, we will make sure that  $fh_i(x) \leq f(x)$  for all  $i$  and for all  $x \in \mathbf{R}^2$  (guaranteeing hypothesis (i)) and also that the support of  $h_i$  is disjoint from  $U_{a_i}$  for some sequence  $a_i \rightarrow \infty$  (guaranteeing hypothesis (ii)).

**Proposition 9.16.** *Let  $U_1 \subset U_2 \subset \dots$  be open subsets of a space  $X$  so that  $X = \bigcup_{i=1}^\infty U_i$ . Let  $h_1, h_2, \dots$  be a sequence of self homeomorphisms of  $X$  satisfying the two hypotheses:*

- (i)  $h_i(U_j) \subseteq U_j$  for all  $i$  and  $j$ .
- (ii) For each  $j$ , the set of  $i$  for which  $h_i|_{U_j} \neq \text{Id}$  is finite.

Then the infinite composition:

$$\cdots \circ h_k \circ h_{k-1} \circ \cdots \circ h_2 \circ h_1$$

is a homeomorphism  $h : X \rightarrow X$ . Further, if  $X$  is a CAT manifold and each  $h_i$  is a CAT isomorphism, then  $h$  is a CAT isomorphism.

*Remark 9.17.* Intuitively, (i) says the  $h_i$ 's “pull in” and (ii) says their supports “move out.”

*Proof of Proposition 9.16.* We may assume that  $h_i|_{U_1} \neq \text{Id}$  for some  $i$  since the general case follows from this special case. For each  $j$ , let  $n(j)$  be the largest positive integer such that  $h_{n(j)}|_{U_j} \neq \text{Id}$ , which exists by hypothesis (ii). Thus:

$$(1) \quad h_i|_{U_j} = \text{Id} \quad \text{for every } i > n(j).$$

As  $U_j \subseteq U_{j+1}$ , we also have:

$$(2) \quad n(j+1) \geq n(j) \quad \text{for every } j.$$

For each  $N \geq n(j)$ , hypothesis (i) along with (1) imply that:

$$(3) \quad (h_{n(j)} \circ \cdots \circ h_1)|_{U_j} = (h_N \circ \cdots \circ h_{n(j)} \circ \cdots \circ h_1)|_{U_j}.$$

We may naturally define  $h : X \rightarrow X$  as follows. Let  $x \in X$ . Then  $x$  lies in  $U_j$  for some  $j$ . We define:

$$(4) \quad h(x) := h_{n(j)} \circ \cdots \circ h_1(x).$$

Properties (2) and (3) show that  $h(x)$  is well defined, independent of alternative choices of  $j$  such that  $x \in U_j$ . Hence, for each  $j$  the restriction  $h|_{U_j}$ , defined by (4), is a homeomorphism onto its image. Therefore,  $h$  is a local homeomorphism, and  $h$  is injective since each given pair of points in  $X$  lies in  $U_j$  for some  $j$ . To conclude  $h$  is a homeomorphism, it remains to show that  $h$  is surjective.

Let  $y \in X$  and choose  $j$  so that  $y \in U_j$ . The homeomorphism:

$$(5) \quad h_{n(j)} \circ \cdots \circ h_1 : X \longrightarrow X$$

sends a unique  $x \in X$  to  $y$ . We claim that  $h(x) = y$ . In the case  $x \in U_j$ , the claim is clear since  $h|_{U_j}$  is defined by (4). Otherwise, choose  $j' > j$  such that  $x \in U_{j'}$ . Then  $n(j') \geq n(j)$  by (2), and:

$$h(x) = h_{n(j')} \circ \cdots \circ h_1(x) = h_{n(j')} \circ \cdots \circ h_{n(j)+1}(y) = y$$

where the third equality holds by (1) since  $y \in U_j$ . We conclude that  $h$  is surjective and  $h$  is a homeomorphism.

For the second conclusion in the proposition, we need only show that  $h$  is a local CAT isomorphism. But this is immediate since if the  $h_i$  are CAT isomorphisms, then for each  $j$  the restriction  $h|_{U_j}$ , defined by (4), is a CAT isomorphism onto its image. This completes the proof of Proposition 9.16.  $\square$

So, to complete the proof of Theorem 9.13, we must show how to do step (1) (a). We will produce diffeomorphisms  $h_i$  satisfying  $f h_i(x) \leq f(x)$  with support in the annulus  $f^{-1}([f(u_0) - 1, \infty))$ , thus guaranteeing the applicability of Proposition 9.16. In particular, for any  $j$  there are only finitely many critical points of  $f|_L$  in the disc of radius  $j + 1$ . After a finite number  $n$  of steps (1) we will have gotten rid of all these critical points (except for those on the boundary of  $L_n$ ), so  $f|_{L_n}$  has no critical points in the disc of radius  $j + 1$  (except for those on the boundary of  $L_n$ ). Consequently, for  $i > n$  we have  $f(u_0) > j + 1$  so the support of  $h_i$  is disjoint from  $U_j$ .

If  $r$  is a component of  $L_i$  and  $a$  and  $b$  are two points of  $r$ , we let  $r[a, b]$  denote the closed segment of  $r$  from  $a$  to  $b$ , oriented going from  $a$  to  $b$ . For convenience, we consider the points in  $\partial L_i$  to be critical points of  $f|_{L_i}$  from here on.

Let  $u_1$  be the local maximum of  $f|_{L_i}$  on which  $f$  assumes the minimum value. Let  $r$  be the component of  $L_i$  containing  $u_1$ . Let  $u_0$  and  $u'_0$  be the critical points of  $f|_{L_i}$  adjacent to  $u_1$  in  $r$ ; the only critical points in  $r[u_0, u'_0]$  are  $u_0, u_1$  and  $u'_0$ . After switching  $u_0$  and  $u'_0$ , if needed, we may assume  $f(u_0) > f(u'_0)$ . Since  $f$  is increasing on  $r[u'_0, u_1]$ , there is a unique  $w_0$  in  $r[u'_0, u_1]$  so that  $f(w_0) = f(u_0)$ .

Let  $A$  denote the complement in  $\mathbf{R}^2$  of the open disk of radius  $f(u_0)$  centered at the origin. Let  $D$  be the compact region in  $A$  bounded by the segment of  $r$  from  $w_0$  to  $u_0$  and an arc of the circle of radius  $f(u_0)$

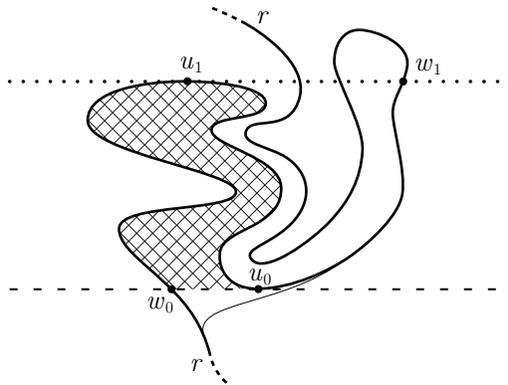


FIGURE 11. Canceling pair of critical points  $u_0$  and  $u_1$ , where  $u_1$  is the least maximum of  $f(x) = |x|$  on the multiray  $L$ . The dashed and dotted lines are arcs of the circles  $|x| = |u_0|$  and  $|x| = |u_1|$ , respectively. The shaded region  $D$  will be pushed below the circle  $|x| = |u_0|$  during the cancellation process. The fine arc indicates the trajectory of the improved ray after cancellation. The least maximum property of  $u_1$  ensures that  $L$  intersects  $D$  in exactly the segment  $r[w_0, u_0] \subset r$ , not more.

between  $u_0$  and  $w_0$  (see Figure 11). We produce  $h_i$  from Assertion 9.18 with  $U$  the complement of the disc of radius  $f(u_0) - 1$ .

**Assertion 9.18.** *For any neighborhood  $U$  of  $D$ , there is a diffeomorphism  $h_i$  of  $\mathbf{R}^2$  so that:*

- (a)  $f(h_i(x)) \leq f(x)$  for all  $x \in \mathbf{R}^2$ .
- (b) The support of  $h_i$  lies in  $U$ .
- (c) The support of  $h_i$  does not intersect  $L_i - r$  and also only intersects  $r$  in a small neighborhood of the segment  $r[w_0, u_0]$ .
- (d) The critical points of the restriction of  $f$  to  $h_i(L_i)$  are the same as those of  $f|_{L_i}$ , except for  $u_1$  and  $u_0$  which are no longer critical or even in  $h_i(L_i)$ .

*Proof of Assertion 9.18.* Note that the interior of  $D$  does not intersect  $L_i$ , since that would give a local maximum of  $f|_{L_i}$  with value  $< f(u_1)$ , contrary to our choice of  $u_1$ . Consequently, we may assume  $U$  does not intersect  $L_i - r$  and does not intersect  $r$  outside a small neighborhood of  $r[w_0, u_0]$ .

We get  $h_i$  by integrating a suitable vector field  $v$  on  $\mathbf{R}^2$ . In particular:

- $v(x) \cdot x \leq 0$  for all  $x \in \mathbf{R}^2$  (to get (a)).
- $v(x) \cdot x = -|x|$  for all  $x$  in some neighborhood  $U'$  of  $D$ .
- $v$  points into  $D$  on the interior of  $r[w_0, u_0]$ .
- The support of  $v$  is contained in  $U$ , does not intersect  $L_i - r$  and does not intersect  $r$  outside of a small neighborhood of  $r[w_0, u_0]$ .

It suffices to find such a  $v$  locally, since then  $v$  is obtained by piecing together with a partition of unity. Finding  $v$  locally is easy. The vector field  $v(x) = -x/|x|$  works on the interior of  $D$ , on the circle of radius  $f(u_0)$  (except possibly at  $w_0$ ), and near  $u_1$ . Near a point  $y \in r[w_0, u_0]$  (with  $y \neq u_1$  and  $y \neq u_0$ ), one may take  $v$  to be  $v' + v''$  divided by the locally positive scalar function  $x \mapsto -(v' + v'') \cdot x/|x|$  where  $v'$  is tangent to  $r$  with  $v' \cdot y < 0$ , and  $v''$  is the unique unit vector at  $y$  directed into  $D$  and tangent at  $y$  to the circle  $f(x) = |y|$ .

Having obtained a vector field  $v$  satisfying the above conditions, one can construct  $h_i$  by elementary methods (cf. [45, pages 10–13]). More precisely, suppose  $g : [a, b] \rightarrow U'$  parameterizes a slightly larger segment of  $r$  than  $r[w_0, u_0]$ . Let  $\phi(x, t)$  be the flow associated to the vector field  $v$ . Note that

$$d/dt(f\phi(x, t)) = v \cdot \nabla f = v(\phi(x, t)) \cdot \phi(x, t)/|\phi(x, t)| = -1$$

as long as  $\phi(x, t) \in U'$ . Consequently,  $f\phi(g(s), t) = fg(s) - t$  as long as  $\phi(g(s) \times [0, t]) \subset U'$ . Since  $v$  enters  $D$  on the interior of  $r[w_0, u_0]$ , we are thus guaranteed that, for some  $\epsilon > 0$ ,  $f\phi(g(s), t) = fg(s) - t$  for all  $t \geq 0$  with  $fg(s) - t \geq f(u_0) - \epsilon$ .

Choose a smooth function  $\alpha : [a, b] \rightarrow [0, \infty)$  with support in  $(a, b)$  so that  $fg - \alpha$  is within  $\epsilon$  of  $f(u_0)$  and has positive derivative everywhere; this is possible since  $f(g(a))$  is slightly less than  $f(u_0)$  and  $f(g(b))$  is slightly greater than  $f(u_0)$ . Let  $\gamma : \mathbf{R} \rightarrow \mathbf{R}$  be a smooth function with compact support such that  $\gamma(0) = 1$ . Then we define  $h_i$  to be the identity outside  $\phi(g([a, b]) \times \mathbf{R})$  and we define  $h_i\phi(g(s), t) = \phi(g(s), t + \alpha(s)\gamma(ct))$  for some appropriate  $c > 0$ . In particular, if  $M_0 = \max \alpha(s)$  and  $M_1 = \min \gamma'(s)$  and  $0 < c < -1/(M_0M_1)$ , then the mapping  $(s, t) \mapsto (s, t + \alpha(s)\gamma(ct))$  is a diffeomorphism (since its restriction to any vertical line has positive derivative), so  $h_i$  is a diffeomorphism. Note that  $h_i(L_i)$  is obtained from  $L_i$  by replacing the segment  $g([a, b])$  with the segment  $\{\phi(g(s), \alpha(s)) \mid s \in [a, b]\}$ .

But  $f\phi(g(s), \alpha(s)) = fg(s) - \alpha(s)$  which has nonzero derivative, so the replacement segment has no critical points of  $f$ . This completes the proof of Assertion 9.18.  $\square$

This completes the proof of the MHLT (Theorem 9.2) for  $m = 2$ .  $\square$

*Remark 9.19.* The basic technique used in the above proof of the 2-MRT (Theorem 9.13) is to ambiently cancel a minimal height local maximum of  $f|_L$  with an adjacent local minimum in a controlled fashion (“2-dimensional embedded Morse theory”). This technique is noteworthy for its simplicity and its utility. The DIFF Schoenflies theorem was nowhere employed as only basic separation properties are needed to obtain the vector field. Indeed, this technique quickly yields proofs of both the DIFF Schoenflies theorem and the DIFF HLT (Theorem 6.1) for  $m = 2$ , the latter without assuming any ray hypothesis, as we now describe.

*Proof of DIFF Schoenflies for  $m = 2$ .* Let  $K$  be a smooth circle in the plane. Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be a coordinate projection. Perturb  $K$  so that  $f$  is Morse on  $K$  with distinct critical values. Let  $m$  and  $M$  be the absolute minimum and maximum points of  $f$  on  $K$ . Now, apply the above technique to the two segments of  $K$  connecting  $m$  and  $M$ . The rest of the proof is an exercise.  $\square$

*Proof of DIFF HLT (Theorem 6.1) for  $m = 2$ .* Let  $N$  be a smooth proper embedding of  $\mathbf{R}^1$  in  $\mathbf{R}^2$ . Let  $P$  be a point in  $N$ . Translate so that  $P$  is the origin in the plane. Push  $N$  to coincide with its tangent line at  $P$  in an  $\varepsilon$ -neighborhood  $N_\varepsilon$  of  $P$  in  $N$ . Perform a homothety so that  $N$  intersects the disk of radius two in a linear segment  $N_2$ . Let  $D^2$  denote the unit disk, and let  $N_1 := N \cap D^2$ . Consider the two component multiray  $L := N - \text{Int } N_1$ . Apply the above cancellation technique to  $L$ , noting that these cancellations fix  $N_2$  pointwise. Finally, apply Lemma 9.15 to  $L \subset \mathbf{R}^2 - \text{Int } D^2$ .  $\square$

*Remark 9.20.* It is natural to consider the  $n$ -dimensional analog of the 2-MRT (Theorem 9.13), namely:

**Theorem 9.21** (Multiray radialization theorem) ( $n$ -MRT). *Let  $L \subset \mathbf{R}^n$  be a smooth proper multiray. If  $n \neq 3$ , then  $L$  is ambient isotopic to a radial multiray.*

Recall that the  $n$ -MRT is ‘false’ in dimension  $n = 3$  because even one proper ray may knot in  $\mathbf{R}^3$  (see Remarks 7.4). On the other hand, if  $n > 3$ , then the  $n$ -MRT holds (any CAT) by the argument in the third paragraph of the proof of the MHLT (Theorem 9.2) for  $m > 3$  and CAT=PL or CAT=DIFF given earlier in this section; for CAT=TOP, this argument uses Homma’s method.

We mention that, for CAT=DIFF and  $n > 3$ , one may prove the  $n$ -MRT via the basic technique used in the above proof of the 2-MRT (Theorem 9.13). Indeed, this approach works with  $\mathbf{R}^n$  replaced by any smooth manifold  $W$  that is collared at infinity. By ray shortening one can assume without loss that  $W = M \times [0, \infty)$ . We claim that  $L$  may be straightened, i.e., there is an ambient isotopy of  $W$  carrying each ray of  $L$  to a ray of the form  $m \times [t, \infty)$ . Since  $n > 3$ , one can slightly perturb  $L$  so that its projection to  $M$  is a one-to-one immersion. This canonically provides a Whitney 2-disk  $D$  for suppression of a pair  $u_1$  and  $u_0$  of critical points, cf. Figure 11; indeed,  $D$  is made up of vertical segments (just two degenerate), and the vector field is vertical. One then concludes as for Theorem 9.13. We need not process the  $u_1$  in min max order but we do need to ensure that  $h_k$  does not increase the  $[0, \infty)$  coordinate, as this guarantees the infinite composition  $\cdots h_k \cdots h_1$  is a diffeomorphism. The interested reader may enjoy seeing where this argument fails in ambient dimension  $n = 3$ ; an infinite number of trefoils tied in a ray reveals the problem (a single trefoil tied in a ray reveals the local problem). One cannot make the projection of  $L$  to  $M$  one to one and thus may no longer exclude  $L$  from the interior of the Whitney disc.

**Two alternative proofs of the MHLT for dimension 2.** We have seen in the proof of the MHLT (Theorem 9.2) in this section that it suffices to give alternative proofs that each closure  $M^2$  in  $\mathbf{R}^2$  of a complementary component of a properly embedded family of lines in  $\mathbf{R}^2$  is isomorphic to a linear gasket. Thus, it suffices to give new proofs of the gasket recognition Theorem 9.3 for dimension 2, that we restate as:

**Theorem 9.22** (2-Gasket recognition theorem (2-GRT)). *Consider a PL 2-manifold  $M$  whose interior is isomorphic to  $\mathbf{R}^2$ , and of which every boundary component is non-compact. Then  $M$  is isomorphic to  $\mathbf{R}^2$ , or to a linear gasket.*

Note that the converse of Theorem 9.22 is trivial.

We pause to offer a broader understanding of this result. We accept as known the following analog for dimension 2 of the Poincaré conjecture:

**Classical fact 9.23** (2-PC). *Every compact 2-manifold  $N^2$  having  $H_1(N; \mathbf{Z}/2\mathbf{Z}) = 0$  is isomorphic to the sphere  $S^2$  or to the disk  $\mathbf{B}^2$ .*

This 2-PC is part of almost any classification of compact PL (or DIFF) surfaces; see for example Section 9 of [28].  $\square$

Aiming to analyze the hypotheses of Theorem 9.22 (2-GRT), we prove:

**Proposition 9.24.** *Consider a connected non-compact 2-manifold  $M^2$ . The following conditions are equivalent:*

- (a)  $\text{Int } M \cong \mathbf{R}^2$ .
- (b)  $M$  is irreducible; in other words every circle PL embedded in  $M$  is the boundary of a PL 2-disk embedded in  $M$ .
- (c)  $M$  is contractible.
- (d)  $H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$ .

*Proof of Proposition 9.24.* Note that all four conditions are invariant under deletion (or addition) of boundary. Thus, without loss of generality, we can and do assume for the proof that  $\partial M = \emptyset$ , i.e.,  $M$  is ‘open.’

We can and do choose to work in the PL category.

By the PL Schoenflies theorem, (a) implies (b). Trivially, (a) implies (c). By (PLCT) in Section 7 of [55], (b) implies (a). By the homotopy axiom for homology, (c) implies (d). To conclude, we prove that (d) implies (b).

Consider any circle  $C$  that is PL embedded in  $M$ . This  $C$  is bicollared, for otherwise its regular neighborhood is a Möbius band, which shows that  $C$  has self-intersection number 1, and hence the class of  $C$  is non-zero in  $H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$ , a contradiction.

Continuing the proof that (d) implies (b), we examine several cases.

*Case 1.*  $C$  does not separate  $M$ . Then another embedded curve  $C'$  in  $M$  exists that intersects  $C$  in a single point and transversally. Thus,  $C$  and  $C'$  have mod 2 intersection number 1 in  $M$ . This shows that the homology classes of  $C$  and  $C'$  in  $H_1(M; \mathbf{Z}/2\mathbf{Z})$  are both nonzero, which contradicts (d). Thus, Case 1 cannot occur.

*Case 2.*  $C$  separates  $M$ . Then, as  $C$  is bicollared, it necessarily cuts  $M$  into two connected pieces,  $M_1$  and  $M_2$ , each with boundary a copy of  $C$ . We now treat two subcases of Case 2 separately.

*Subcase (i).* Neither piece  $M_i$  is compact. Seeking a contradiction, suppose this subcase occurs. There then exists a properly embedded path  $C'$  in  $M$  that intersects  $C$  in a single point and transversally. There is thus a nonzero mod 2 intersection number of  $C$  with  $C'$  proving that the class of  $C$  in  $H_1(M; \mathbf{Z}/2\mathbf{Z})$  is nonzero, a contradiction. Thus this subcase cannot occur. We conclude that the following must always occur.

*Subcase (ii).* One piece, say  $M_1$ , is compact. Then we claim that  $H_1(M_1; \mathbf{Z}/2\mathbf{Z}) = 0$ . To prove this claim, suppose the contrary. Capping  $M_1$  with a 2-disk  $B$  yields a PL closed 2-manifold  $N_1$  with

$$H_1(N_1; \mathbf{Z}/2\mathbf{Z}) \cong H_1(M_1; \mathbf{Z}/2\mathbf{Z}) \neq 0.$$

In  $H_1(N_1; \mathbf{Z}/2\mathbf{Z})$ , Poincaré duality provides a pair of compact curves  $C_1$  and  $C'_1$  (disjoint from  $B$  by general position) having non-zero intersection number mod 2. They lie in both  $M$  and  $M_1$  and have the same non-zero intersection number in  $M$  as in  $M_1$ , contradicting  $H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$ . This proves the claim.

Next, since  $H_1(M_1; \mathbf{Z}/2\mathbf{Z}) = 0$ , the classical 2-PC tells us that  $M_1$  is a 2-disk. This proves for  $M$  the irreducibility condition (b), and thereby completes the proof of Proposition 9.24.  $\square$

By the above Proposition 9.24, the following assertion is equivalent to 2-GRT.

**Assertion 9.25.** *Every noncompact contractible 2-manifold  $M^2$  is isomorphic to a linear gasket in  $\mathbf{H}^2$ , or to  $\mathbf{H}^2$  itself.*

To conclude, we present two quite different proofs of this assertion.

*Sketch of a classical topological proof of Assertion 9.25.* We can

assume  $\text{CAT} = \text{TOP}$ . The case of the assertion where  $M$  has 3 boundary components readily implies it for 2, 1 or 0 boundary components, so we assume  $M$  has  $\geq 3$  boundary components.

We shall use a pleasant  $\text{TOP}$  classification of such  $M^2$  stated below. It is an easy consequence of three difficult classical theorems applied to the double  $DM$  of  $M$  formed from two copies of  $M$  with their boundaries identified. More details and references are given in [56].

The first theorem was discovered by Schoenflies [52] and states that a compact connected subset  $J$  of the plane is a circle if and only if its complement has two components and each point of  $J$  is accessible as the unique limit of a path in each. The second is the Osgood-Schoenflies theorem (proved circa 1912, see [55]) stating that every circle  $J$  topologically embedded in the plane bounds a topological 2-disk. The third is due to K erekjart o [33] and classifies *all* surfaces *without* boundary, in particular  $DM$ , in terms of what is now known as the (K erekjart o-Freudenthal) *end compactification*.

**Classification 9.26.** *The end compactification of a noncompact contractible surface  $M$  with nonempty boundary, written  $E(M) = M \cup e(M)$ , is always a 2-disk, whose interior is  $\text{Int } M$ , and whose boundary circle  $\partial E(M)$  is the disjoint union  $\partial M \cup e(M)$  where  $e(M)$  is the compact and totally disconnected end space of  $M$ . Thus,  $M$  is homeomorphic to a 2-disk  $E(M)$  minus a compact part  $e(M)$  of its boundary.*

*Proof of Classification (in outline).* By [33], the end compactification  $E(DM)$  is  $S^2$ . Then [52] shows that the obvious involution  $\tau$  on  $E(DM)$  has fixed point set a Jordan curve, and finally the Osgood-Schoenflies theorem shows that  $S^2/\tau = E(M)$  is a 2-disk as required.  $\square$

The remainder of the proof of Assertion 9.25 is elementary. Identify  $E(M)$  to the round Euclidean disk  $\mathbf{B}^2 \subset \mathbf{R}^2$  and consider the convex hull  $\text{Hull}(e(M))$  in  $\mathbf{R}^2$ . Since  $M$  has  $\geq 3$  ends, the convex hull  $\text{Hull}(e(M))$  is topologically a 2-disk in  $\mathbf{R}^2$ , and all its extremal points (as a convex subset of  $\mathbf{R}^2$ ) constitute  $e(M) \subset \partial \mathbf{B}^2$ . Hence, there is a standard homeomorphism  $\text{Hull}(e(M)) \rightarrow \mathbf{B}^2$ , respecting every ray emanating from the barycenter of the hull, and fixing  $e(M)$ . Thus,  $M$  itself is  $\text{TOP}$  isomorphic to the linear gasket

$$\text{Hull}(e(M)) \cap \text{Int } \mathbf{B}^2 = \text{Hull}(e(M)) \cap \mathbf{H}^2. \quad \square$$

*Sketch of a geometric proof of Assertion 9.25.* There is a famous procedure that tiles any closed 2-manifold  $M_g$  of genus  $g \geq 2$  by compact hexagonal 2-cells (tiles), and then constructs a hyperbolic structure for  $M_g$  in which each 2-cell has geodesic edges and all vertex angles  $\pi/2$ . In reply to our inquiry about known geometric proofs, J.-P. Otal promptly suggested that a similar approach would prove the assertion.

The case of the assertion for  $\geq 3$  boundary components implies the general case, so *we restrict to this case in what follows.*

We work in the DIFF category.

Given an arbitrary enumeration of the components of  $\partial M$  (called *sides* below), there is a construction procedure of ‘cut and paste’ topology to construct on  $M$  a DIFF tiling in which each two-dimensional tile is closed and is either a compact hexagonal tile or a noncompact cusp tile (a triangle with one ideal vertex at Alexandroff’s infinity).

These tiles will fit together as follows. Each finite vertex lies in  $\partial M$ . Each hexagonal tile  $H$  has 3 of its 6 edges alternatively in three distinct sides of  $\partial M$ , and the remainder of  $\partial H$  lies in  $\text{Int } M$ . The intersection of any hexagonal tile with any distinct tile is either empty or a common edge joining distinct components of  $\partial M$ . Every cusp tile meets  $\partial M$  in its two infinite sides while its compact side is shared with one hexagonal tile. The nerve of the tiling of  $M$  is thus a tree  $T$  with one trivalent vertex for each hexagonal tile and one univalent vertex for each cusp tile.

The procedure is initialized by construction of a hexagonal tile that meets the first three sides in the given enumeration of sides. After the first three sides, for each successive new side, one more hexagonal tile  $H$  is inductively constructed;  $H$  meets the new side and those two of the earlier sides that are in a topological sense adjacent. This induction completes the construction of all the hexagonal tiles. To terminate the tiling procedure, the cusp tiles are then defined to be the closures of the components of the complement of the union of all the hexagonal tiles. The cusp tiles correspond bijectively to the isolated ends of  $M$ .

This DIFF tiling is well defined by the given enumeration of the sides of  $M$ , up to a DIFF isomorphism of tilings that is piecewise DIFF isotopic to the identity of  $M$ .

Each tile has a hyperbolic structure with the length of each compact edge equal to one, and a right angle at each vertex (infinity excepted). After an isotopy of such structures, they fit together to form a complete hyperbolic structure  $\sigma$  on  $M$  making  $\partial M$  geodesic.

This hyperbolic structure  $\sigma$  on  $M$  is well defined by the tiling, up to isometry ambient isotopic to the identity.

To conclude, one develops  $M_\sigma$  isometrically into  $\mathbf{H}^2$ , proceeding inductively tile by tile, climbing up the above tree  $T$ , to realize  $M$  as a linear gasket in  $\mathbf{H}^2$ .  $\square$

*Remark 9.27.* The hyperbolic structure  $\sigma$  on  $M$  obtained by the above tiling procedure is often distinct from any structure obtained by the classical proof; indeed, for every isolated end of  $M$  the limit points of its cusp tile neighborhood in the ideal circle at infinity  $\partial\mathbf{B}^2$  of  $\mathbf{H}^2$  constitute a whole compact interval rather than a point. However, this clear geometric distinction can be suppressed as follows: the cut-locus in  $M_\sigma$  of  $\partial M_\sigma$  is a properly embedded piecewise geodesic graph  $\Gamma \subset \text{Int } M$ , which meets each tile in a standard way. The convex hull of the closure of  $\Gamma$  in  $\mathbf{B}^2$ , intersected with  $\mathbf{H}^2$ , is a smaller but visibly diffeomorphic copy  $M'$  of  $M$  whose hyperbolic structure is of the sort obtained in the classical proof.

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