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SINGLE RATIONAL ARCTANGENT IDENTITIES FOR π

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1. Introduction. The most common methods of calculating π to large numbers of decimal places utilize an infinite sum for the arctangent function. All of these infinite sums converge faster when the argument is small. I first began looking at rational arctangent identities for π in 1991. Except for $\arctan(1) = \pi/4$, all rational arctangent identities for π use two distinct angles (e.g., $\arctan(1/2) + \arctan(1/3) = \pi/4$). I wondered why there was no known identity of the form $n * \arctan(x) = \pi$, where n is a large natural number and $|x|$ is a small rational number, and whether such an identity existed. The first major step towards generalizing these identities came in October, 1995, when I independently discovered the pattern preceding Theorem 2. This solved the problem for $\pi/4$ only, however. I was convinced I could solve the problem for all rational multiples of π. The next breakthrough came in September 1998 (at a bus stop no less). The final piece of the puzzle fell into place. This paper contains both of these ideas and all supporting details.

2. Single Rational Arctangent Identities for π. We are interested in identities for π that determine a rational multiple of π with only one evaluation of the arctangent function where the argument is rational.

DEFINITION 1. *A single rational arctangent identity for π is any identity of the form $n\arctan(x) = k\pi$ where n is natural, $x \neq 0$ is rational and k is an integer.*

It follows from the definition that $k \neq 0$ since $n, x \neq 0$. Clearly every identity of the form $\frac{n}{m}\arctan(x) = \frac{a}{b}\pi$, where n, m, b are natural, $a \neq 0$ is an integer, and $x \neq 0$ is rational, reduces to a single rational arctangent identity for π, so we need only generalize the latter. First, we derive a useful expression for $\tan(n\arctan(x))$ where $n = 0, 1, \ldots$ and x is real. Recalling that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

we get:

$$\begin{aligned}
\tan(0 * \arctan(x)) &= \frac{0}{1} \\
\tan(1 * \arctan(x)) &= \frac{x}{1} \\
\tan(2 * \arctan(x)) &= \frac{2x}{1-x^2} \\
\tan(3 * \arctan(x)) &= \frac{x + \frac{2x}{1-x^2}}{1 - x\frac{2x}{1-x^2}} = \frac{3x - x^3}{1-3x^2} \\
\tan(4 * \arctan(x)) &= \frac{x + \frac{3x-x^3}{1-3x^2}}{1 - x\frac{3x-x^3}{1-3x^2}} = \frac{4x-4x^3}{1-6x^2+x^4}
\end{aligned}$$

It appears that $\tan(n\arctan(x)) = p_n(x)/q_n(x)$, where $p_n(x)$ is the sum of odd power terms in binomial expansion with alternating signs, and $q_n(x)$ is the sum of even

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power terms in binomial expansion with alternating signs. That is:

$\left\lfloor \frac{n-1}{2} \right\rfloor$.

$$p_n(x) = \sum_{i=0}^{\lfloor n/2\rfloor - 1} (-1)^i \binom{n}{2i+1} x^{2i+1} \text{ and } q_n(x) = \sum_{i=0}^{\lfloor n/2\rfloor} (-1)^i \binom{n}{2i} x^{2i}, \tag{1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (the binomial coefficients).

Before we prove that equations (1) hold, recall that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, a fundamental ingredient of the Pascal triangle, which can be easily proven by induction [2].

THEOREM 2. *$\tan(n \arctan(x))$ is defined by equations (1) for all natural n and real x.*

Proof. We will proceed by induction on n. We have seen that equations (1) hold true for $n = 1, 2, 3, 4$. Assume (1) is true for all $n \leq k$. We must show (1) is true for $k+1$. There are two cases to consider: k even and k odd. Suppose k is even. Then $k+1$ is odd. So

$$\begin{aligned}
\tan((1+k)\arctan(x)) &= \frac{x + \frac{p_k(x)}{q_k(x)}}{1 - x\frac{p_k(x)}{q_k(x)}} \text{“=”} \frac{xq_k(x) + p_k(x)}{q_k(x) - xp_k(x)} \\
&= \frac{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} x^{2i+1} + \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} x^{2i+1}}{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} x^{2i} - \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} x^{2i+2}} \\
&= \frac{\sum_{i=0}^{\frac{k}{2}-1} (-1)^i \left[\binom{k}{2i} + \binom{k}{2i+1}\right] x^{2i+1} + (-1)^{\frac{k}{2}} x^{k+1}}{1 + \sum_{i=0}^{\frac{k}{2}-1} (-1)^{i+1} \left[\binom{k}{2i+2} + \binom{k}{2i+1}\right] x^{2i+2}} \\
&= \frac{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i+1} x^{2i+1}}{\sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i} x^{2i}},
\end{aligned}$$

as desired. The proof for k odd is virtually identical, as the reader may wish to verify. □

Note that Theorem 2 applies to all nonzero integral n, since if $n < 0$ then $-n > 0$ and $\tan(n\arctan(x)) = -\tan(-n\arctan(x)) = -p_{-n}(x)/q_{-n}(x)$.

It is transparent from Theorem 2 that $p_n(x)$ and $q_n(x)$ are rational for all integral n and rational x. Clearly then, $\tan(n\arctan(x))$ is rational for rational x provided $q_n(x) \neq 0$.

We now apply Theorem 2 to $\pi/4$.

THEOREM 3. *If we have $n\arctan(x) = \pi/4$ for some nonzero integral n, then the only possible rational values for x are $x = \pm 1$.*

Proof. $n\arctan(x) = \frac{\pi}{4} \Rightarrow \tan(n\arctan(x)) = 1 \Rightarrow p_n(x)/q_n(x) = 1 \Rightarrow p_n(x) = q_n(x) \Rightarrow q_n(x) - p_n(x) = 0 \Rightarrow 1 - \binom{n}{1}x - \binom{n}{2}x^2 + \cdots \pm \binom{n}{n-1}x^{n-1} \pm x^n = 0.$

The only possible rational roots of this polynomial are $x = \pm 1$ by the rational root theorem. $\square$

Theorem 3 characterizes all single rational arctangent identities for $\pi/4$.

Before we characterize all single rational arctangent identities for all rational multiples of π, it is necessary that we make some observations. First, we look at $p_n(1)$ and $q_n(1)$. From (1) we get: $p_1(1) = 1$, $p_2(1) = 2$, $p_3(1) = 2$, $p_4(1) = 0$, $q_1(1) = 1$, $q_2(1) = 0$, $q_3(1) = -2$ and $q_4(1) = -4$.

Further inspection leads one to the following conjecture (where $n = 4d+r$, $r < 4$):

(2)

$$p_n(1) = \begin{cases} 0, & n \equiv 0 (\text{mod } 4) \\ (-4)^d, & n \equiv 1 (\text{mod } 4) \\ 2(-4)^d, & n \equiv 2 (\text{mod } 4) \\ 2(-4)^d, & n \equiv 3 (\text{mod } 4) \end{cases} \quad \text{and } q_n(1) = \begin{cases} (-4)^d, & n \equiv 0 (\text{mod } 4) \\ (-4)^d, & n \equiv 1 (\text{mod } 4) \\ 0, & n \equiv 2 (\text{mod } 4) \\ -2(-4)^d, & n \equiv 3 (\text{mod } 4) \end{cases}.$$

Proving equations (2) hold is straightforward after we prove the next proposition.

PROPOSITION 4. *$p_{k+1}(1) = q_k(1) + p_k(1)$ and $q_{k+1}(1) = q_k(1) - p_k(1)$.*

Proof. Case 1. (k is even) Then $k+1$ is odd and

$$\begin{aligned} p_{k+1}(1) &= \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i+1} = \sum_{i=0}^{\frac{k}{2}} (-1)^i \left[\binom{k}{2i+1} + \binom{k}{2i} \right] \\ &= \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} + \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} \\ &= p_k(1) + q_k(1). \end{aligned}$$

Also,

$$\begin{aligned} q_{k+1}(1) &= \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k+1}{2i} = \sum_{i=0}^{\frac{k}{2}} (-1)^i \left[\binom{k}{2i} + \binom{k}{2i-1} \right] \\ &= \sum_{i=0}^{\frac{k}{2}} (-1)^i \binom{k}{2i} - \sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k}{2i+1} \\ &= q_k(1) - p_k(1). \end{aligned}$$

Case 2. (k is odd) The proof is virtually identical to Case 1, as the reader may wish to verify. $\square$

THEOREM 5. *Equations (2) hold for all natural n.*

Proof. We will proceed by induction on n. We have seen that equations (2) hold true for $n = 1, 2, 3, 4$. Assume equations (2) hold for all $n \leq k$ and write $k = 4 * d + r$, $r < 4$.

Case 1. $k \equiv 0(\text{mod}4)$ implies $p_k(1) = 0$ and $q_k(1) = (-4)^d$. Proposition 4 implies $p_{k+1}(1) = (-4)^d$ and $q_{k+1}(1) = (-4)^d$.

Case 2. $k \equiv 1(\text{mod}4)$ implies $p_k(1) = (-4)^d$ and $q_k(1) = (-4)^d$. Proposition 4 implies $p_{k+1}(1) = 2(-4)^d$ and $q_{k+1}(1) = 0$.

Case 3. $k \equiv 2(\text{mod}4)$ implies $p_k(1) = 2(-4)^d$ and $q_k(1) = 0$. Proposition 4 implies $p_{k+1}(1) = 2(-4)^d$ and $q_{k+1}(1) = -2(-4)^d$.

Case 4. $k \equiv 3(\bmod 4)$ implies $p_k(1) = 2(-4)^d$ and $q_k(1) = -2(-4)^d$. Proposition 4 implies $p_{k+1}(1) = 0$ and $q_{k+1}(1) = (-4)^{d+1}$. □

Now, we notice that for a single rational arctangent identity for π $n\arctan(x) = k\pi$ if and only if $0 = \tan(n\arctan(x)) = p_n(x)/q_n(x)$, which occurs if and only if $p_n(x) = 0$ and $q_n(x) \neq 0$. We are inclined to conjecture that if $p_n(x) = 0$ and x is rational then $x = 0$ or ± 1, however, it is not at all clear how to prove this straightaway for all natural n. The key is to first prove it is true for all prime $n > 0$.

THEOREM 6. *If we have $n > 0$ prime and x rational such that $p_n(x) = 0$ then $x = 0$ or ± 1.*

Proof. We first prove the result for $n = 2$ and then for all $n > 2$. Assume $n = 2$, then $0 = p_2(x) = 2x \Rightarrow x = 0$ as desired. Now, assume $n > 2$ and prime, then n is odd. So,

$$
\begin{aligned}
p_n(x) = 0 &\Rightarrow \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{2i+1} x^{2i+1} = 0 \\
&\Rightarrow nx - \binom{n}{3}x^3 + \binom{n}{5}x^5 - \cdots \mp \binom{n}{n-2}x^{n-2} \pm x^n = 0 \\
&\Rightarrow x = 0 \text{ or } n - \binom{n}{3}x^2 + \binom{n}{5}x^4 - \cdots \mp \binom{n}{n-2}x^{n-3} \pm x^{n-1} = 0 \\
&\Rightarrow x = \pm 1 \text{ or } \pm n
\end{aligned}
$$

by the rational root theorem since n is prime. Suppose $x = \pm n$ is a root, then $n \pm \binom{n}{3}n^2 \mp \cdots \pm n^{n-1} = 0 \Rightarrow 1 \pm \binom{n}{3}n^1 \mp \cdots \pm n^{n-2} = 0$, but the only possible rational roots of this polynomial are $\pm 1 \Rightarrow n = \pm 1$. But this contradicts the assumption that $n > 2$. Hence, $x = 0$ or ± 1. □

Next, we show that we can solve the problem for $n\arctan(x) = a\pi$, where n is composite, by "stepping down" by prime factors of n.

THEOREM 7. *If $n > 1$ natural and x is rational such that $\tan(n\arctan(x)) = 0$ then $x = 0$ or ± 1.*

Proof. n has a prime factorization, say $n = s_1 s_2 \cdots s_j$, where each $s_i > 1$ and the s_i's are not necessarily distinct. We will proceed by induction on j. First assume $j = 1$. If x is a rational such that $\tan(s_1 \arctan(x)) = 0$ then $p_{s_1}(x) = 0$ and $q_{s_1}(x) \neq 0$ and $x = 0$ or ± 1 by Theorem 6. Now, assume true for all $j < h$. Let n be some natural such that $n = s_1 s_2 \cdots s_h$, where each $s_i > 1$ and the s_i's are not necessarily distinct. Further, let x be some rational such that $\tan(n\arctan(x)) = 0$ and write $\phi = s_2 \cdots s_h \arctan(x)$. Then we have:

$$
0 = \tan(n\arctan(x)) = \tan(s_1 s_2 \cdots s_h \arctan(x)) = \tan(s_1\phi) = \frac{p_{s_1}(\tan\phi)}{q_{s_1}(\tan\phi)},
$$

which implies that $p_{s_1}(\tan\phi) = 0$ and $q_{s_1}(\tan\phi) \neq 0$. Now, $(s_2 \cdots s_h)$ is a natural number and x is rational, so $\tan(\phi)$ is defined by equations (1) by Theorem 2. Hence, $\tan(\phi)$ is either rational or undefined. If $\tan(\phi)$ is undefined then $p_{s_1}(\tan\phi) \neq 0$, a contradiction. Therefore, $\tan(\phi)$ is rational, say $y = \tan(\phi)$. Then we have $p_{s_1}(y) = 0$ where y is rational and $s_1 > 1$ is prime, so $y = 0$ or ± 1 by Theorem ~~1.5~~ 6. Suppose $y = \pm 1$, this implies that $\pm 1 = y = \tan(\phi) = \tan(s_2 \cdots s_h \arctan(x))$. But then $x = \pm 1$ by the proof of ~~Proposition 4.~~ Thm 3. Therefore, suppose $y = 0$, this implies $0 = y = \tan(\phi) = \tan(s_2 \cdots s_h \arctan(x))$, where $s_2 \cdots s_h$ is a natural number composed of $h - 1 < h$ primes. Hence, $x = 0$ or ± 1 by the inductive hypothesis. □

Now, we combine our results and ~~generalize~~ characterize all single rational arctangent identities for π.

THEOREM 8. *The following are equivalent:*
(i) $n\arctan(x) = a\pi$ *is a single rational arctangent identity for* π.
(ii) $x = \pm 1$ *and* $n \equiv 0(\mathrm{mod}\ 4)$, *where* n *is natural.*

Proof. First, we prove (i) $\Rightarrow$ (ii). Assume (i) is true. Then by definition n is natural, $x \neq 0$ is rational and $a \neq 0$ is an integer. Then $n\arctan(x) = a\pi \Rightarrow \tan(n\arctan(x)) = 0$ and $x = \pm 1$ by Theorem7 ($x \neq 0$ by definition). So, we have $\tan(n\arctan(\pm 1)) = 0 \Rightarrow \tan(n\arctan(1)) = 0 \Rightarrow p_n(1) = 0$. By equations (2), we have that $n \equiv 0(\mathrm{mod}\ 4)$.

Now we prove (ii) $\Rightarrow$ (i).

Assume (ii) is true. If $x = -1$, then $p_n(x) = p_n(-1) = -p_n(1)$ and $q_n(x) = q_n(-1) = q_n(1)$ by equations (1). Furthermore, $p_n(1) = 0$ and $q_n(1) \neq 0$ by equations (2), which implies that $\tan(n\arctan(x)) = \dfrac{p_n(x)}{q_n(x)} = \dfrac{p_n(\pm 1)}{q_n(\pm 1)} = \dfrac{\pm p_n(1)}{q_n(1)} = 0$. Hence, $n\arctan(x) = \arctan(0) = k\pi$ for some integer k. Since $\arctan(x) = \arctan(\pm 1) = \pm\pi/4$, we have that $\pm n\pi/4 = k\pi \Rightarrow k = \pm n/4 \neq 0$. □

3. Conclusion. We have seen that there does not exist a single rational arctangent identity for π, hence for any rational multiple of π, that converges faster than $\arctan(1) = \pi/4$. To obtain faster convergence using rational arctangent identities, one must make at least two distinct arctangent evaluations. The logical continuation would be to generalize identities for π that use two arctangent evaluations.

After the completion of the work presented here, it was found that Gauss had done just that, but only for $\pi/4$, $2\pi/4$, and $3\pi/4$. Gauss' method is outlined in Wrench [4], the key relation being:

$$\arctan(x) = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right) \tag{3}$$

However, using equation (3) to prove the result of this paper only leads one to the polynomials in (1). To see this, suppose $n\arctan(x) = a\pi$ is a single rational arctangent identity for π. Equation (3) implies that $\dfrac{1}{2i}\ln\left(\left(\dfrac{1+ix}{1-ix}\right)^n\right) = a\pi$. Now, we use the fact that:

$$\tan(\phi) = \frac{1}{i}\frac{e^{2i\phi}-1}{e^{2i\phi}+1} = z \Rightarrow e^{2i\phi} = \frac{1+iz}{1-iz},$$

to see that:

$$e^{\ln\left(\left(\frac{1+ix}{1-ix}\right)^n\right)} = \frac{1+i(\tan(a\pi))}{1-i(\tan(a\pi))} = 1.$$

Simplifying gives us $\left(\dfrac{1+ix}{1-ix}\right)^n = 1$, which implies that $(1+ix)^n - (1-ix)^n = 0$. The Binomial Theorem and some simplification shows that $(1+ix)^n - (1-ix)^n = 0$ is exactly equivalent to $p_n(x) = 0$. It should be mentioned that even the best rational arctangent identities for π have their limitations. Using logarithms it is easy to see that each iteration of the Gregory series for $\arctan(x)$, where $|x| < 1$ and $x \neq 0$, yields approximately $|2\log_{10}(x)|$ more digits accuracy, which is linear in $\log_{10}(x)$. There are recursive formulas for π based on elliptic integrals that have quadratic, cubic, quadruple, and septet convergence rates. The interested reader may refer to Kanada [1].

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In Memoriam.

A. D. Stewart. Pi Mu Epsilon notes with sadness the passing of A. D. Stewart, Councilor from 1978-1984 from Prairie View A & M University in Prairie View, Texas. A. D. Stewart was a gentle man who could always be counted upon as a valuable presence at Pi Mu Epsilon meetings. His loosely knotted tie and warm smile were always recognizable characteristics. Prof. Stewart encouraged his students from Prairie View A&M to be active mathematically on campus and to take part in the national PME meetings. He was a loyal ΠΜΕ supporter and will be very much missed.

Robert G. Kane. Professor Robert G. Kane, Associate Professor of Mathematics and Computer Science at the University of Detroit Mercy (Michigan Beta chapter), died July 12 at age 65. Professor Kane was the moderator of Michigan Beta chapter, Pi Mu Epsilon, for many years. He taught mathematics at the University of Detroit (pre-merger) since 1957.