ARTIN PRESENTATIONS, TRIANGLE GROUPS, AND 4-MANIFOLDS

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Dedicated to the memory of Elmar Winkelnkemper.

Abstract. González-Acuña showed that Artin presentations characterize closed, orientable 3-manifold groups. Winkelnkemper later discovered that each Artin presentation determines a smooth, compact, simply-connected 4-manifold. We utilize triangle groups to find all Artin presentations on two generators that present the trivial group. We then determine all smooth, closed, simply-connected 4-manifolds with second betti number at most two that appear in Artin presentation theory.

1. Introduction

An *Artin presentation* is a group presentation $r = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$ such that the following holds in the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$

$$x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n)$$

González-Acuña [Gon75, Thm. 4] showed that every closed, orientable 3-manifold admits an open book decomposition with planar page. As a corollary, he obtained the following algebraic characterization of 3-manifold groups.

Theorem (González-Acuña [Gon75, Thm. 6])**.** *A group G is the fundamental group of a closed, orientable 3-manifold if and only if G admits an Artin presentation r for some n.*

Winkelnkemper [Win02, p. 250] discovered that each Artin presentation r determines not only a closed, orientable 3-manifold $M^3(r)$ but also a smooth, compact, simply-connected 4-manifold $W^4(r)$ such that $\partial W^4(r) = M^3(r)$. All intersection forms are represented by some $W^4(r)$ [Win02, pp. 248–250]. If $M^3(r)$ is the 3-sphere, then we consider the smooth, *closed*, simply-connected 4-manifold $X^4(r) = W^4(r) \cup_\partial D^4$ obtained from $W^4(r)$ by closing up with a 4-handle.

While all closed, orientable 3-manifolds appear in Artin presentation theory, it is unknown which 4-manifolds appear as a $W^4(r)$ or an $X^4(r)$. The only contractible manifold $W^4(r)$ is D^4 (when $r = \langle | \rangle$ is the empty Artin presentation). So, no Mazur manifold appears as a $W^4(r)$. Nevertheless, there are no known smooth, closed, simply-connected 4-manifolds that do not appear as an $X^4(r)$; many interesting closed 4-manifolds are known to appear this way including all elliptic surfaces $E(n)$ where $E(2)$ is diffeomorphic to the Kummer surface $K3$ [CW04].

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We determine all closed 4-manifolds $X^4(r)$ where r is an Artin presentation on two generators. (For $n = 0$ and $n = 1$, the problem is straightforward: only S^4, $\mathbb{C}P^2$, and $\overline{\mathbb{C}P^2}$ appear.) Theorem 4.2 gives the complete list of these manifolds: $\mathbb{C}P^2 \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $\overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$, and $S^2 \times S^2$. Exotic simply-connected, closed 4-manifolds are currently not known to exist with second betti number ≤ 2. Theorem 4.2 shows that such manifolds, whether or not they exist in general, do not appear in Artin presentation theory. Exotic 4-manifolds do appear in Artin presentation theory with second betti number ≥ 10 [Cal08]. We conjecture that closed, exotic 4-manifolds appear in Artin presentation theory with second betti number three, and that this relates to the Torelli subgroup in Artin presentation theory (see [Win02, p. 250] and [Cal08]).

Our proof of Theorem 4.2 uses the classification of Artin presentations on two generators and properties of classical triangle groups to find all Artin presentations on two generators that present the trivial group. We then introduce a move on Artin presentations on two generators which preserves the 4-manifolds. Using this move and the Kirby calculus, we then identify the 4-manifolds. It would be interesting to find other such moves in Artin presentation theory. Armas-Sanabria has shown certain three generator Artin presentations present nontrivial groups [Arm12].

For each nonnegative integer n, let $\mathcal{R}_n$ denote the set of Artin presentations on n generators. We include a proof of the folklore theorem (see [Win02, p. 245] and [Gon75, p. 10]) that $\mathcal{R}_n$ is a group isomorphic to the product $P_n \times \mathbb{Z}^n$ of the pure braid group P_n with the rank n free abelian group.

Throughout, $X \approx Y$ means that X is orientation preserving diffeomorphic to Y. If X is an oriented manifold, then $\overline{X}$ denotes the same manifold with the opposite orientation.

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2. Artin Presentations

In this section, we review fundamental properties of Artin presentations and fix notation. We begin by recalling how each Artin presentation arises naturally from a homeomorphism of a compact 2-disk with holes.

Let Ω_n denote the compact 2-disk with n holes as in Figure 2.1. The boundary com-

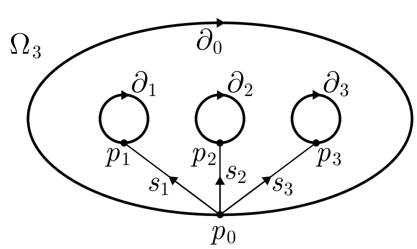

Figure 2.1. Compact 2-disk with n holes denoted Ω_n (the case $n = 3$ is depicted).

ponents $\partial_0, \partial_1, \ldots, \partial_n$ of Ω_n are parameterized clockwise and are based at $p_0, p_1, \ldots, p_n$ respectively. For each $1 \leq i \leq n$, let s_i be an oriented segment from p_0 to p_i as in Figure 2.1. Given a path α, let $\overline{\alpha}$ denote the reverse path and let $[\alpha]$ denote the path homotopy class of α. Concatenation of paths—performed left to right—and the induced operation on classes will be denoted by juxtaposition. For each $1 \leq i \leq n$, let $x_i = [s_i \partial_i \overline{s_i}]$ as in Figure 2.2. So, $\pi_1(\Omega_n, p_0) \cong F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is free of rank n.

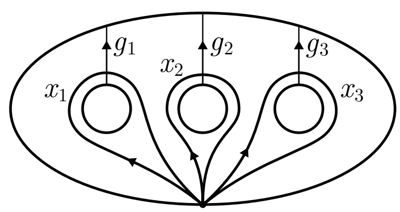

FIGURE 2.2. Generators x_i of $\pi_1(\Omega_3, p_0)$.

Let $h : \Omega_n \to \Omega_n$ be a homeomorphism that equals the identity (point-wise) on the boundary of Ω_n. Then, the induced homomorphism $h_\sharp : \pi_1(\Omega_n, p_0) \to \pi_1(\Omega_n, p_0)$ is an automorphism. For each $1 \leq i \leq n$, define $r_i = [s_i(h \circ \overline{s_i})] \in F_n$. Define the presentation $r = r(h) = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$.

Claim 2.1. *For each $1 \leq i \leq n$, we have $h_\sharp(x_i) = r_i^{-1} x_i r_i$.*

Proof. Note that $h \circ \overline{s_i} = \overline{h \circ s_i}$, $r_i^{-1} = [(h \circ s_i)\overline{s_i}]$, and $h \circ \partial_i = \partial_i$. Therefore

$$\begin{aligned} h_\sharp(x_i) &= [h \circ (s_i \partial_i \overline{s_i})] = [h \circ s_i]\,[h \circ \partial_i]\,[h \circ \overline{s_i}] \\ &= [h \circ s_i]\,[\overline{s_i}]\,[s_i]\,[\partial_i]\,[\overline{s_i}]\,[s_i]\,[h \circ \overline{s_i}] \\ &= r_i^{-1} x_i r_i \end{aligned}$$

□

Claim 2.2. *The presentation $r = r(h)$ determined by h is an Artin presentation.*

Proof. The following holds in F_n

$$\begin{aligned} x_1 x_2 \cdots x_n &= [\partial_0] = [h \circ \partial_0] \\ &= h_\sharp(x_1 x_2 \cdots x_n) = h_\sharp(x_1) h_\sharp(x_2) \cdots h_\sharp(x_n) \\ &= (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n) \end{aligned}$$

where the last equality used Claim 2.1. □

Let $\mathrm{Homeo}(\Omega_n, \partial\Omega_n)$ denote the group of homeomorphisms of Ω_n that equal the identity (point-wise) on $\partial\Omega_n$. By Claim 2.2, we have a function

$$\psi : \mathrm{Homeo}(\Omega_n, \partial\Omega_n) \to \mathcal{R}_n$$

given by $\psi(h) = r(h)$. If h and h' are isotopic relative to $\partial\Omega_n$, then $\psi(h) = \psi(h')$. We will show that ψ is a surjective homomorphism of groups. First, we present a few examples. Given an Artin presentation $r \in \mathcal{R}_n$, we define $\pi(r)$ to be the group presented by r and $A(r)$ to be the exponent sum matrix of r meaning $[A(r)]_{ij}$ equals the exponent sum of x_i in r_j. Note that $A(r)$ is an $n \times n$ integer matrix, the abelianization of $\pi(r)$ is isomorphic to $\mathbb{Z}^n/\mathrm{Im}A$ where $\mathrm{Im}A$ denotes the image of $A : \mathbb{Z}^n \to \mathbb{Z}^n$, and $\pi(r)$ is perfect if and only if $A(r)$ is unimodular (that is, $\det A = \pm 1$).

Examples 2.3.

(1) The empty presentation $\varepsilon = \langle | \rangle$ is the unique Artin presentation in $\mathcal{R}_0$. Here, $\pi(\varepsilon)$ is trivial and $A(r) = []$ is empty.

(2) The Artin presentations associated to T_c and T_c^{-1} in Figure 2.3 are $\langle x_1 \mid x_1 \rangle$ and $\langle x_1 \mid x_1^{-1} \rangle$ respectively. Given a homeomorphism $h \in \mathrm{Homeo}\,(\Omega_n, \partial\Omega_n)$, one computes

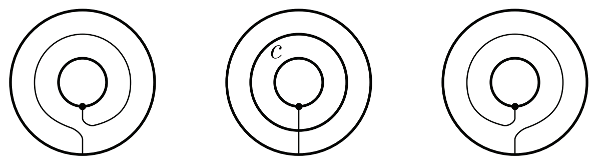

FIGURE 2.3. Simple closed curve c in Ω_1 (center), result of right Dehn twist T_c about c (right), and result of left Dehn twist T_c^{-1} about c (left).

the associated Artin presentation $r = \psi(h)$ as follows. Recall Figures 2.1 and 2.2. The relation r_i is obtained by starting at p_i, following $h(\overline{s_i})$, and recording x_j (respectively x_j^{-1}) each time g_j is crossed from left to right (respectively right to left).

(3) Each $r \in \mathcal{R}_1$ has the form $r = \langle x_1 \mid x_1^a \rangle$ for some integer a. Here, $\pi(r) \cong \mathbb{Z}/\,|a|\,\mathbb{Z}$ and $A(r) = [a]$.

(4) Let $a_1, a_2, \ldots, a_n$ be any integers. Then, $r = \langle x_1, x_2, \ldots, x_n \mid x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n} \rangle$ is an Artin presentation in $\mathcal{R}_n$ that presents the free product of cyclic groups $\mathbb{Z}/\,|a_i|\,\mathbb{Z}$.

(5) The Artin presentations associated to T_c and T_c^{-1} in Figure 2.4 are $\langle x_1, x_2 \mid x_1x_2, x_1x_2 \rangle$ and $\langle x_1, x_2 \mid x_2^{-1}x_1^{-1}, x_2^{-1}x_1^{-1} \rangle$ respectively.

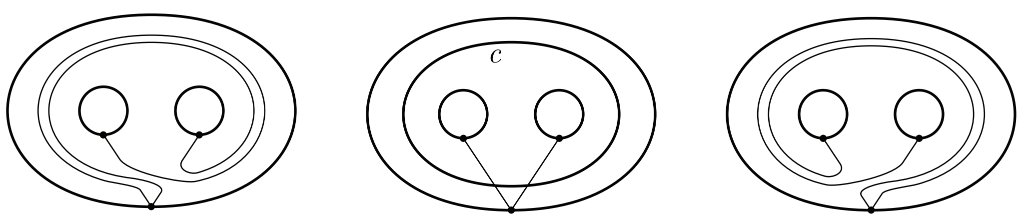

FIGURE 2.4. Simple closed curve c in Ω_2 (center), result of right Dehn twist T_c about c (right), and result of left Dehn twist T_c^{-1} about c (left).

(6) Each $r = \langle x_1, x_2 \mid r_1, r_2 \rangle \in \mathcal{R}_2$ has the form $r_1 = x_1^{a-c}(x_1x_2)^c$ and $r_2 = x_2^{b-c}(x_1x_2)^c$ for some integers a, b, and c (see [Cal07, p. 360] and [Win02, p. 245]). We denote such an Artin presentation by $r(a, b, c)$, so $A(r(a, b, c)) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$. For instance, $r(-1, -3, 2)$ presents the binary icosahedral group $I(120)$. We mention that $\pi(r(a, b, c)) \cong \pi(r(-a, -b, -c))$ by the map $x_i \mapsto x_i$.

Remark 2.4. For each n, $\mathcal{R}_n$ has a natural binary operation. Namely, let $r, u \in \mathcal{R}_n$. Define the composition $u \circ r$ to be the presentation $t = \langle x_1, x_2, \ldots, x_n \mid t_1, t_2, \ldots, t_n \rangle$ as follows: let R_i be obtained by substituting $u_j^{-1}x_ju_j$ for x_j in r_i, and define $t_i = u_iR_i$.

We now show that $t \in \mathcal{R}_n$ *provided* $r, u \in \operatorname{Im}\psi$ (shortly, we will see that $\operatorname{Im}\psi = \mathcal{R}_n$, which means this operation is defined on all of $\mathcal{R}_n$). Let $h, k \in \operatorname{Homeo}(\Omega_n, \partial\Omega_n)$ such that $\psi(h) = r$ and $\psi(k) = u$. For each $1 \leq j \leq n$, Claim 2.1 implies that $h_\sharp(x_j) = r_j^{-1} x_j r_j$ and $k_\sharp(x_j) = u_j^{-1} x_j u_j$. So, for each $1 \leq i \leq n$, we have

$$\begin{aligned}(k \circ h)_\sharp (x_i) &= k_\sharp (h_\sharp(x_i)) = k_\sharp \left(r_i^{-1} x_i r_i\right) \\ &= k_\sharp(r_i)^{-1} u_i^{-1} x_i u_i k_\sharp(r_i) \\ &= (u_i k_\sharp(r_i))^{-1} x_i (u_i k_\sharp(r_i)) \\ &= (u_i R_i)^{-1} x_i (u_i R_i) \\ &= t_i^{-1} x_i t_i\end{aligned}$$

Thus

$$x_1 x_2 \cdots x_n = (k \circ h)_\sharp (x_1 x_2 \cdots x_n) = (t_1^{-1} x_1 t_1)(t_2^{-1} x_2 t_2) \cdots (t_n^{-1} x_n t_n)$$

and so $t = u \circ r \in \mathcal{R}_n$. Summarizing, $\psi| : \operatorname{Homeo}(\Omega_n, \partial\Omega_n) \to \operatorname{Im}\psi$ is surjective, the domain is a group, the codomain is a set with a binary operation, and ψ respects the operations—meaning $\psi(k \circ h) = \psi(k) \circ \psi(h)$. These facts imply that $\operatorname{Im}\psi$ is a group and $\psi|$ is a surjective homomorphism. Note that each Artin presentation in the examples above lies in $\operatorname{Im}\psi$. The identity in $\operatorname{Im}\psi$ is $\langle x_1, x_2, \ldots, x_n \mid 1, 1, \ldots, 1\rangle$.

Let D_n denote the compact 2-disk with $n \geq 0$ marked points $q_1, q_2, \ldots, q_n$ in its interior as in Figure 2.5. We choose q_i to lie at the center of the simple closed curve ∂_i and let

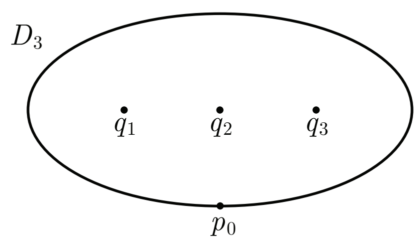

FIGURE 2.5. Surface D_n that is the compact 2-disk D^2 with n marked points $q_1, q_2, \ldots, q_n$ in its interior (the case $n = 3$ is depicted).

$Q = \{q_1, q_2, \ldots, q_n\}$. Let $D^2 - Q$ be the 2-disk with $n \geq 0$ punctures.

Claim 2.5. *The function* ψ *is surjective. Hence,* $\psi : \operatorname{Homeo}(\Omega_n, \partial\Omega_n) \to \mathcal{R}_n$ *is a surjective homomorphism of groups.*

Proof. Let $t \in \mathcal{R}_n$. It suffices to prove that $t \in \operatorname{Im}\psi$ since then the second conclusion follows from Remark 2.4. Define the endomorphism $\beta : F_n \to F_n$ by $x_i \mapsto t_i^{-1} x_i t_i$. By Artin[1] [Art25, pp. 64–68], the map β is a pure braid group automorphism of F_n. So, there exists a homeomorphism h' of $D^2 - Q$ that is a product of homeomorphisms corresponding to braid group generators such that: h' is the identity on ∂D^2, h' sends each puncture to itself (by purity of β), and $h'_\sharp = \beta$. Further, we can and do assume that the restriction h of h' to Ω_n equals the identity on $\partial\Omega_n$. Hence, $h \in \operatorname{Homeo}(\Omega_n, \partial\Omega_n)$ and $h_\sharp = \beta$. Let

[1]Artin's beautiful algebraic argument appeared originally in German and later in English [Art47, pp. 114–115]; Birman gave an exposition of Artin's argument in her book [Bir75, Thm. 1.9, p. 30].

$r = \psi(h) \in \mathcal{R}_n$. For each $1 \leq i \leq n$, $h_\sharp(x_i) = r_i^{-1} x_i r_i$ by Claim 2.1. So, $t_i^{-1} x_i t_i = r_i^{-1} x_i r_i$ and $x_i \left(t_i r_i^{-1}\right) = \left(t_i r_i^{-1}\right) x_i$. Commuting elements in F_n are powers of the same word [MKS76, p. 42]. Thus, $t_i r_i^{-1} = x_i^{a_i}$ for some integer a_i, which implies $t_i = x_i^{a_i} r_i$. Define $u = \langle x_1, x_2, \ldots, x_n \mid x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n} \rangle \in \mathcal{R}_n$. By the examples above, there exists $k \in \operatorname{Homeo}(\Omega_n, \partial\Omega_n)$ such that $\psi(k) = u$. As $u_j^{-1} x_j u_j = x_j$ for each $1 \leq j \leq n$, we have $t = u \circ r \in \operatorname{Im} \psi$ as desired. □

Remark 2.6. Let $\operatorname{Mat}(n, \mathbb{Z})$ be the additive group of $n \times n$ integer matrices. The function $A : \mathcal{R}_n \to \operatorname{Mat}(n, \mathbb{Z})$ is a homomorphism—meaning $A(u \circ r) = A(u) + A(r)$. To see this, note that $u \circ r \in \mathcal{R}_n$ by Claim 2.5. By definition, $[A(u \circ r)]_{ij}$ equals the exponent sum of x_i in $u_j R_j$ where R_j is obtained from r_j by substituting $u_k^{-1} x_k u_k$ for x_k. Therefore, $[A(u \circ r)]_{ij} = [A(u)]_{ij} + [A(r)]_{ij}$ as desired.

Let $\operatorname{Homeo}_0(\Omega_n, \partial\Omega_n)$ be the subgroup of $\operatorname{Homeo}(\Omega_n, \partial\Omega_n)$ consisting of homeomorphisms (fixed point-wise on $\partial\Omega_n$) that are isotopic relative to $\partial\Omega_n$ to the identity.

Claim 2.7. *The kernel of ψ equals $\operatorname{Homeo}_0(\Omega_n, \partial\Omega_n)$.*

Before we prove Claim 2.7, we need a technical lemma. Incidentally, this lemma is the reason we may—and typically do—assume the individual relations r_i in each Artin presentation r are freely reduced (see also [Cal07, pp. 363–365]).

Lemma 2.8. *Let h be a diffeomorphism of Ω_n fixed point-wise on $\partial\Omega_n$. Assume that the arcs $h(\overline{s_i})$ meet the segments g_j (see Figure 2.2) in general position. Suppose some $h(\overline{s_i})$ crosses some g_j ($j = i$ is possible) at the point a and then, without crossing any other segment g_k, $h(\overline{s_i})$ crosses g_j in the opposite direction at the point b as in Figure 2.6. Then,*

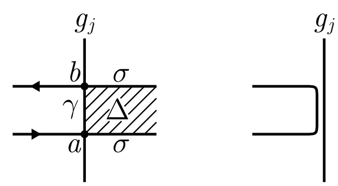

FIGURE 2.6. Arc $h(\overline{s_i})$ meeting g_j consecutively in opposite directions without meeting any segments g_k in-between (left) and result of isotopy pushing σ across Δ to a parallael copy of γ (right).

there is an isotopy of h relative to $\partial\Omega_n$ that eliminates the crossings a and b. This isotopy introduces no new crossings between the arcs $h(\overline{s_k})$ and g_l, though it may eliminate other crossings (in case some $h(\overline{s_k})$ meets γ).

Proof of Lemma 2.8. Let γ and σ be the arcs between a and b in g_j and $h(\overline{s_i})$ respectively. So, $C = \gamma \cup \sigma$ is a simple closed curve in $\operatorname{Int} \Omega_n$. By the Schoenflies theorem, C bounds a 2-disk $\Delta \subseteq \operatorname{Int} D^2$. We show Δ is disjoint from $\partial\Omega_n$. As $\Delta \subseteq \operatorname{Int} D^2$, Δ is disjoint from ∂_0. Suppose, by way of contradiction, that Δ meets ∂_k for some $1 \leq k \leq n$. Then, $\partial_k \subseteq \operatorname{Int} \Delta$ by connectedness. If $k \neq j$, then the segment g_k must meet C by the Jordan Curve Theorem JCT. Hence, g_k meets σ which contradicts the hypotheses on $h(\overline{s_i})$. Next, assume $k = j$. Observe that the segment g_j begins on ∂_j in $\operatorname{Int} \Delta$. Also, an open segment of g_j ending at the lower point a or b (a in Figure 2.6) does not lie in Δ. Therefore, g_j must meet σ at a point other than a or b by the JCT. Again, this contradicts the hypotheses on $h(\overline{s_i})$.

Therefore, $\Delta \cap \partial\Omega_n = \emptyset$ and $\Delta \subseteq \operatorname{Int}\Omega_n$. The isotopy is now obtained by pushing σ across Δ to a parallel copy of γ as in Figure 2.6. $\square$

Proof of Claim 2.7. Let $h \in \ker\psi$, and let $r = \psi(h)$. By an isotopy of h relative to $\partial\Omega_n$, we can and do assume h is a diffeomorphism and the arcs $h(\overline{s_i})$ meet the segments g_j in general position. As $h \in \ker\psi$, each $r_i = 1$ which means r_i freely reduces to 1. Let $x_j^{\pm1}x_j^{\mp1}$ be adjacent letters in some r_i ($j = i$ is possible); this corresponds to $h(\overline{s_i})$ crossing some g_j and then, without crossing any other segment g_k, crossing g_j in the opposite direction. By Lemma 2.8, an isotopy of h relative to $\partial\Omega_n$ eliminates these crossings. Repeating this operation finitely many times, we may assume that the arcs $h(\overline{s_i})$ are disjoint from the segments g_j. By a small isotopy of h relative to $\partial\Omega_n$, we may assume height restricts to a Morse function on the arcs $h(s_i)$ with distinct critical values. The minimal local maximum of height on $h(s_1)$ may be ambiently cancelled with an adjacent local minimum (see [CKS12, pp. 1845–1852], especially Figure 11). Repeating this operation finitely many times, the arc $h(s_1)$ has strictly increasing height. By integrating an appropriate horizontal vector field on Ω_n, we may assume $h(s_1) = s_1$. Integrating a vector field tangent to s_1, we may assume h also equals the identity on s_1. One may repeat this procedure on s_2—without disturbing s_1—and so forth. Thus, we have h is a diffeomorphism of Ω_n equal to the identity on $\partial\Omega_n$ and on the segments s_i for $1 \leq i \leq n$. Cutting Ω_n open along the arcs s_i, h becomes a homeomorphism of the 2-disk equal to the identity on boundary. By Alexander's trick [FM12, 47–48], this homeomorphism is isotopic to the identity relative to boundary. This latter isotopy induces an isotopy relative to $\partial\Omega_n$ of h to the identity. $\square$

The mapping class group of Ω_n is

$$\operatorname{Mod}(\Omega_n) = \operatorname{Homeo}(\Omega_n, \partial\Omega_n) / \operatorname{Homeo}_0(\Omega_n, \partial\Omega_n)$$

For useful equivalent definitions of $\operatorname{Mod}(\Omega_n)$, see [FM12, pp. 44–45]. Claims 2.5 and 2.7 immediately imply the following.

Corollary 2.9. *The function* $\operatorname{Mod}(\Omega_n) \to \mathcal{R}_n$ *given by* $[h] \mapsto \psi(h)$ *is an isomorphism.*

Recall from Figure 2.5 above that D_n is the compact 2-disk with n marked points in its interior. The mapping class group of D_n is

$$\operatorname{Mod}(D_n) = \operatorname{Homeo}(D_n, \partial D_n) / \operatorname{Homeo}_0(D_n, \partial D_n)$$

where homeomorphisms and isotopies fix $\partial D_n = S^1$ point-wise and the marked points may be permuted [FM12, p. 45]. The pure mapping class group $\operatorname{PMod}(D_n)$ is the subgroup of $\operatorname{Mod}(D_n)$ that fixes each marked point individually [FM12, pp. 90]. Let B_n denote the classical n strand braid group and P_n denote the n strand pure braid group. There are canonical isomorphisms $B_n \cong \operatorname{Mod}(D_n)$ and $P_n \cong \operatorname{PMod}(D_n)$ [FM12, pp. 243–249]. Recall that $D^2 - Q$ is the 2-disk with n punctures. One may regard marked points as punctures (see [FM12, p. 45]) in the sense that there are canonical isomorphisms $\operatorname{Mod}(D_n) \cong \operatorname{Mod}\left(D^2 - Q\right)$ and $\operatorname{PMod}(D_n) \cong \operatorname{PMod}\left(D^2 - Q\right)$.

Claim 2.10. *There is a canonical isomorphism* $\operatorname{Mod}(\Omega_n) \cong P_n \times \mathbb{Z}^n$.

Proof. We will define the homomorphisms in the sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}^n \arrow[r, "\delta"] & \operatorname{Mod}(\Omega_n) \arrow[r, "\eta|"] \arrow[l, dashed, bend left, "\mu"] & \operatorname{PMod}(D_n) \arrow[r] & 1 \end{tikzcd} \tag{2.1}$$

and show the sequence is exact and left split. For each $1 \le i \le n$, let T_i denote a right Dehn twist about a simple closed curve in Ω_n parallel to ∂_i as in Figure 2.3. If $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, then define $\delta(a) = [T_1^{a_1} T_2^{a_2} \cdots T_n^{a_n}]$. As these Dehn twists commute, δ is a homomorphism. The map μ is defined to be the composition: first the isomorphism in Corollary 2.9, second the homomorphism A (see Remark 2.6 above), and third return the diagonal. As each of these three maps is a homomorphism, μ is a homomorphism. Observe that $\mu \circ \delta = \mathrm{id}$ on $\mathbb{Z}^n$. So, δ is injective and (2.1) is exact on the left.

There is a canonical homomorphism $\eta : \mathrm{Mod}\,(\Omega_n) \to \mathrm{Mod}\,(D_n)$ induced by the inclusion $\Omega_n \subseteq D_n$ [FM12, pp. 82–84]. By [FM12, Thm. 3.18, p. 84] (or [FM12, Prop. 3.19, p. 85] for $n = 1$), $\ker \eta = \langle [T_1], [T_2], \ldots, [T_n] \rangle_{\mathrm{FA}} \cong \mathbb{Z}^n$ where FA indicates free abelian group. By the definition of $\mathrm{Mod}\,(\Omega_n)$, $\mathrm{Im}\,\eta \subseteq \mathrm{PMod}\,(D_n)$ and we have the restriction homomorphism $\eta| : \mathrm{Mod}\,(\Omega_n) \to \mathrm{PMod}\,(D_n)$. As $\ker \eta = \ker \eta|$, the sequence (2.1) is exact in the middle. Let $[h] \in \mathrm{PMod}\,(D_n)$. By an isotopy relative to ∂D_n and Q, we may assume h also equals the identity on and inside each ∂_i for $1 \le i \le n$. If h' is the restriction of h to Ω_n, then $\eta|([h']) = [h]$. So, $\eta|$ is surjective and (2.1) is exact on the right.

Therefore, (2.1) is short exact and left split. By [DF04, Prop. 26, p. 385], there is an induced isomorphism $\mathrm{Mod}\,(\Omega_n) \xrightarrow{\cong} \mathrm{PMod}\,(D_n) \times \mathbb{Z}^n \cong P_n \times \mathbb{Z}^n$. $\square$

Corollary 2.9 and Claim 2.10 prove the following folklore theorem.

Theorem 2.11. *For each $n \ge 0$, there are canonical isomorphisms*

$$\mathcal{R}_n \cong \mathrm{Mod}\,(\Omega_n) \cong P_n \times \mathbb{Z}^n$$

Examples 2.12. Figure 2.7 depicts five framed pure braids corresponding (from left to right) to: the Dehn twists T_c^{-1} and T_c from Figure 2.3, the Dehn twists T_c^{-1} and T_c from Figure 2.4, and the Artin presentation $r(a, b, c)$.

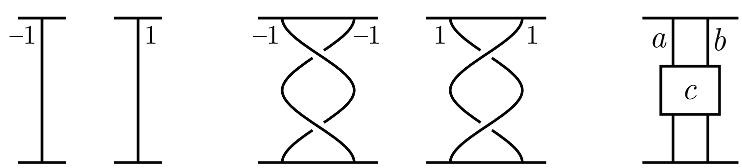

FIGURE 2.7. Framed pure braids.

We now shift our attention to 3- and 4-manifolds in Artin presentation theory. Each Artin presentation $r \in \mathcal{R}_n$ determines the following.

$\pi(r)$	the group presented by r
$A(r)$	the exponent sum matrix of r
$h(r)$	a self-diffeomorphism of Ω_n
$M^3(r)$	a closed, oriented 3-manifold
$W^4(r)$	a smooth, compact, simply-connected, oriented 4-manifold

By Corollary 2.9, r determines a self-diffeomorphism $h = h(r)$ of Ω_n equal to the identity on $\partial\Omega_n$ and unique up to isotopy relative to $\partial\Omega_n$. The 3-manifold $M^3(r)$ is defined by

Winkelnkemper's open book construction with planar page Ω_n (see González-Acuña [Gon75] and Winkelnkemper [Win02]). Namely, consider the mapping torus $\Omega(h)$ of h which is obtained from $\Omega_n \times [0,1]$ by identifying $(x,1)$ with $(h(x),0)$ for each $x \in \Omega_n$. The boundary of $\Omega(h)$ equals $(\partial\Omega_n) \times S^1$, and $M^3(r)$ is obtained from $\Omega(h)$ by gluing on $(\partial\Omega_n) \times D^2$ using the identity function on $(\partial\Omega_n) \times S^1$. The fundamental group of $M^3(r)$ is isomorphic to $\pi(r)$ (see [Gon75, p. 10] or [Win02, p. 247]). In particular, $M^3(r)$ is an integer homology 3-sphere if and only if $A(r)$ is unimodular. Using the symplectic property of closed surface homeomorphisms, Winkelnkemper observed that $A(r)$ is always symmetric for an Artin presentation r (see [Win02, p. 250], or see [Cal07] for an algebraic proof of this fact). This led Winkelnkemper to discover that r determines a 4-manifold using a sort of relative open book construction as follows. Embed Ω_n in S^2, and let C be the closure in S^2 of the complement of Ω_n (so, C is a disjoint union of $n+1$ smooth 2-disks). Extend h to S^2 then to D^3, and let H be the resulting self-diffeomorphism of D^3. The mapping torus $W(H)$ of H contains $C \times S^1$ in its boundary. Then, $W^4(r)$ is obtained from $W(H)$ by gluing on $C \times D^2$ in the canonical way. In particular, $\partial W^4(r) = M^3(r)$ and the intersection form of $W^4(r)$ is given by $A(r)$. If $M^3(r)$ is the 3-sphere, then we define $X^4(r) = W^4(r) \cup_\partial D^4$ a smooth, closed, simply-connected, oriented 4-manifold (that is, we close up with a 4-handle). By Cerf's theorem [Cer68], a 4-handle may be added in an essentially unique way, and so $X^4(r)$ is well-defined.

An alternative definition of $W^4(r)$ is as follows (see also [CW04, §2]). Let $r \in \mathcal{R}_n$ be an Artin presentation. By Theorem 2.11, r determines an integer framed pure braid. The framing of the ith strand is $[A(r)]_{ii}$. Let $L(r)$ be the framed pure link in $S^3 = \partial D^4$ obtained as the closure of this framed pure braid. Define $W^4(r)$ to be D^4 union n 2-handles attached along $L(r)$. So, $L(r)$ is a Kirby diagram for $W^4(r)$ (see [GS99, p. 115] for an introduction to Kirby diagrams). Figure 2.8 gives Kirby diagrams for $W^4(r)$ where $r \in \mathcal{R}_1$ and $r \in \mathcal{R}_2$. For example, $X^4(\langle x_1 \mid x_1 \rangle) \approx \mathbb{C}P^2$ and $X^4(\langle x_1 \mid x_1^{-1} \rangle) \approx \overline{\mathbb{C}P^2}$.

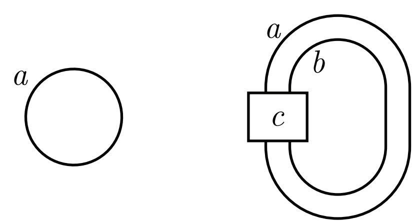

FIGURE 2.8. Kirby diagrams for $W^4(r)$ where $r = \langle x_1 \mid x_1^a \rangle \in \mathcal{R}_1$ (left) and $r = r(a,b,c) \in \mathcal{R}_2$ (right).

Remark 2.13. Three basic diffeomorphisms between the 4-manifolds $W^4(r(a,b,c))$ are as follows where a, b, and c are any integers.

$$W^4(r(a,b,c)) \approx W^4(r(b,a,c)) \tag{2.2}$$

$$W^4(r(a,b,1)) \approx W^4(r(a,b,-1)) \tag{2.3}$$

$$W^4(r(-a,-b,-c)) \approx \overline{W^4(r(a,b,c))} \tag{2.4}$$

The first two diffeomorphisms are given by simple isotopies of the Kirby diagrams: for (2.2) interchange the two link components, and for (2.3) flip one component to switch the sign of the single twist. Regarding (2.3), we mention that while the links are not equivalent as *oriented* links, the resulting 4-manifolds are independent of the orientations of the link components. The third diffeomorphism is a special case of the fact that given a Kirby diagram for a 2-handlebody Y, one obtains a Kirby diagram for $\overline{Y}$ by switching all crossings (that is, take a mirror of the link) and multiplying each framing coefficient by -1. If $M^3(r)$ is S^3, then all three diffeomorphisms also hold with X^4 in place of W^4.

3. Triangle Groups and Artin Presentations

We recall basic facts about triangle groups (for details, see Magnus [Mag74, Ch. II] and Ratcliffe [Rat06, §7.2]). Let l, m, and n be integers greater than or equal to 2. Let $\Delta = \Delta(l,m,n)$ be a triangle with angles π/l, π/m, and π/n. Define

$$\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n}$$

The triangle Δ is: *spherical* and lies in $X = S^2$ if $\delta > 1$, *Euclidean* and lies in $X = \mathbb{R}^2$ if $\delta = 1$, and *hyperbolic* and lies in $X = \mathbb{H}^2$ if $\delta < 1$. The *triangle reflection group* $T^*(l,m,n)$ is the group generated by the reflections of X in the lines containing the sides of Δ. The *triangle group* $T(l,m,n)$ (sometimes called a *von Dyck group*) is the index 2 subgroup of $T^*(l,m,n)$ consisting of orientation preserving isometries of X. Geometrically, $T(l,m,n)$ is generated by the rotations of X about the vertices of Δ by $2\pi/l$, $2\pi/m$, and $2\pi/n$ respectively. The triangle group $T(l,m,n)$ is presented by $\langle x, y \mid x^l, y^m, (xy)^n \rangle$. Notice that $T(l,m,n)$ is independent up to isomorphism of the order in which the integers l, m, and n are listed. The triangle group $T(l,m,n)$ is: *spherical* and finite (but nontrivial) if $\delta > 1$, *Euclidean* and infinite if $\delta = 1$, and *hyperbolic* and infinite if $\delta < 1$. For example, $T(2,3,5)$ is the *icosahedral group* isomorphic to the order 60 alternating group A_5 on five letters. The infinite groups $T(3,3,3)$ and $T(3,3,4)$ correspond respectively to triangular tilings of the Euclidean and hyperbolic planes.

Lemma 3.1. *Let $r = r(a,b,c) \in \mathcal{R}_2$. If $|a-c|$, $|b-c|$, and $|c|$ are all greater than or equal to 2, then $\pi(r)$ is nontrivial. If in addition $1/|a-c| + 1/|b-c| + 1/|c| \leq 1$, then $\pi(r)$ is infinite.*

Proof. We construct a surjective group homomorphism $\pi(r) \twoheadrightarrow T(|a-c|, |b-c|, |c|)$. Add the relation $(x_1x_2)^c$ to r to obtain

$$\begin{aligned}
\pi(r) &\twoheadrightarrow \left\langle x_1, x_2 \mid x_1^{a-c}(x_1x_2)^c, x_2^{b-c}(x_1x_2)^c, (x_1x_2)^c \right\rangle \\
&\cong \left\langle x_1, x_2 \mid x_1^{a-c}, x_2^{b-c}, (x_1x_2)^c \right\rangle \\
&\cong \left\langle x_1, x_2 \mid x_1^{|a-c|}, x_2^{|b-c|}, (x_1x_2)^{|c|} \right\rangle \\
&= T(|a-c|, |b-c|, |c|)
\end{aligned}$$

Now, apply properties of triangle groups recalled above. □

Examples 3.2. Consider the groups $\pi(r(-1,-3,2))$ and $\pi(r(10,1,3))$. Both groups are perfect since their exponent sum matrices are unimodular. Lemma 3.1 implies that $\pi(r(-1,-3,2))$ is nontrivial and $\pi(r(10,1,3))$ is infinite. The proof of Lemma 3.1 shows that $\pi(r(-1,-3,2))$ surjects onto $T(3,5,2) \cong A_5$.

Theorem 3.3. *Let $r = r(a,b,c) \in \mathcal{R}_2$. If $\pi(r)$ is trivial, then the 3-tuple (a,b,c) lies in the following list where $-(a,b,c) = (-a,-b,-c)$.*

(3.1) $(\pm 1, \pm 1, 0)$ *(four 3-tuples)*

(3.2) $\pm(2, 1, \pm 1)$ *and* $\pm(1, 2, \pm 1)$ *(eight 3-tuples)*

(3.3) $\pm(1, 5, 2)$*,* $\pm(5, 1, 2)$*,* $\pm(2, 5, 3)$*,* $\pm(5, 2, 3)$ *(eight 3-tuples)*

(3.4) $(a, 0, \pm 1)$ *and* $(0, b, \pm 1)$ *where* $a, b \in \mathbb{Z}$

(3.5) $(c \pm 1, c \mp 1, c)$ *where* $c \in \mathbb{Z}$

Proof. As $\pi(r)$ is trivial, $A(r)$ must be unimodular which means $ab - c^2 = \pm 1$. Now, the basic idea is that either $|c| \leq 1$ is small and $ab = c^2 \pm 1$ determines a and b, or $|c| > 1$ is larger and Lemma 3.1 forces a or b to be close to c. We have $ab = c^2 \pm 1$ and, by Lemma 3.1, $|a - c| \leq 1$, $|b - c| \leq 1$, or $|c| \leq 1$. Notice that (a, b, c) appears in the given list if and only if $-(a, b, c)$ appears. Indeed, as $\pi(r(a, b, c)) \cong \pi(r(-a, -b, -c))$ (see Examples 2.3(6)), our list must have this property. So, it suffices to assume $c \geq 0$ for the rest of the proof. If $c = 0$, then $ab = \pm 1$, which gives the tuples (3.1). If $c = 1$, then $ab = 0$ or $ab = 2$. The former gives the tuples (3.4), and the latter gives the tuples (3.2).

Assume now that $c > 1$. Then, $|a - c| \leq 1$ or $|b - c| \leq 1$, and so a or b equals $c - 1$, c, or $c + 1$. If $a = c$, then $cb = c^2 \pm 1$ implies that $c| \pm 1$, a contradiction. Similarly, $b \neq c$. Thus, $a = c \pm 1$ or $b = c \pm 1$.

Case 1: $ab = c^2 + 1$**.** Suppose $a = c \pm 1$. Then, $ab = c^2 + 1$ implies $a | c^2 + 1$, and $a = c \pm 1$ implies $a | c^2 - 1$. So, $a | 2$ and, as $a = c \pm 1$ and $c > 1$, we have $a = 1$ or $a = 2$. This gives the tuples $(1, 5, 2)$ and $(2, 5, 3)$. Similarly, $b = c \pm 1$ gives the tuples $(5, 1, 2)$ and $(5, 2, 3)$.

Case 2: $ab = c^2 - 1$**.** Then, $a = c \pm 1$ if and only if $b = c \mp 1$. This gives the tuples (3.5). □

Remark 3.4. It is not difficult to verify the converse of Theorem 3.3 directly using Tietze transformations. This converse also follows from the Kirby calculus arguments in the next section. Hence, Theorem 3.3 lists exactly the Artin presentations on two generators that present the trivial group.

4. 4-manifolds

In this section, we show that $M^3(r)$ is S^3 for each r listed in Theorem 3.3, and we identify the corresponding closed 4-manifolds $X^4(r) = W^4(r) \cup_\partial D^4$. First, we present a useful operation.

Lemma 4.1. *Let* $r = r(a, b, c) \in \mathcal{R}_2$*. There are diffeomorphisms*

$$W^4(r(a, b, c)) \approx W^4(r(a + b - 2c, b, b - c)) \tag{†}$$

$$W^4(r(a, b, c)) \approx W^4(r(a, a + b - 2c, a - c)) \tag{‡}$$

In particular, the corresponding 3*-manifolds* $M^3(r)$ *are diffeomorphic, and the corresponding groups* $\pi(r)$ *are isomorphic.*

Proof. For the first diffeomorphism, proceed as shown in Figure 4.1. In the second diagram in Figure 4.1, the middle circle is parallel to the b-framed circle and has linking number b with it. If the a- and b-framed circles are oriented clockwise, then the indicated 2-handle slide is a handle subtraction; the framing of the a-framed circle changes to $a + b - 2c$ (see [GS99, p. 141]). The isotopy of the middle diagram in Figure 4.1 is explained in Figure 4.2. The

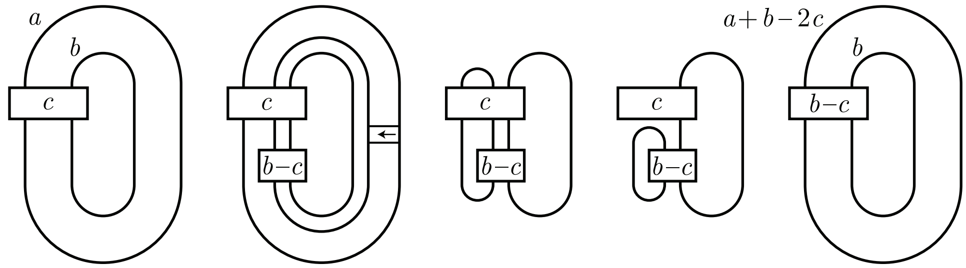

FIGURE 4.1. From the left: Kirby diagram for $W^4(r(a,b,c))$, a 2-handle slide, and results of two isotopies.

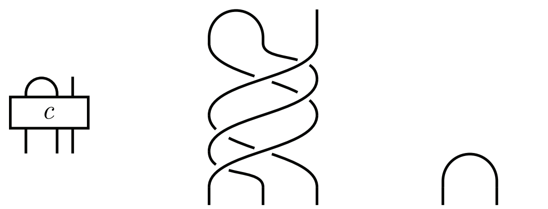

FIGURE 4.2. From the left: a portion of the middle diagram in Figure 4.1, the same portion (enlarged) with $c = 1$ and without box notation, and the result of the isotopy of the portion.

result of Figure 4.1 is a Kirby diagram for $W^4(r(a+b-2c, b, b-c))$. For (‡), instead slide the b-framed circle over the a-framed circle in a similar manner. The remaining claims in the lemma follow from (†) and (‡) by taking boundaries. $\square$

Theorem 4.2. *For each Artin presentation r listed in Theorem 3.3, $M^3(r)$ is S^3. Furthermore, the corresponding closed 4-manifolds $X^4(r)$ are as follows.*

(4.1) $X^4(r(1,1,0)) \approx \mathbb{C}P^2 \# \mathbb{C}P^2$ *and* $X^4(r(1,-1,0)) \approx \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

(4.2) $X^4(r(2,1,1)) \approx \mathbb{C}P^2 \# \mathbb{C}P^2$

(4.3) $X^4(r(5,1,2)) \approx X^4(r(5,2,3)) \approx \mathbb{C}P^2 \# \mathbb{C}P^2$

(4.4) $X^4(r(a,0,1)) \approx \begin{cases} S^2 \times S^2 & \text{if } a \text{ is even} \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{if } a \text{ is odd} \end{cases}$

(4.5) $X^4(r(c+1,c-1,c)) \approx \begin{cases} S^2 \times S^2 & \text{if } c \text{ is odd} \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{if } c \text{ is even} \end{cases}$

The 4-manifolds for the remaining 3-tuples in Theorem 3.3 are determined immediately from those just listed and Remark 2.13.

Proof. First, (4.1) is clear since the Kirby diagrams are two-component unlinks with framings ± 1. By (†), $W^4(r(2,1,1)) \approx W^4(r(1,1,0))$, and (4.2) now follows from (4.1). Next, $W^4(r(2,1,-1)) \approx W^4(r(2,1,1))$ by Remark 2.13, $W^4(r(2,1,-1)) \approx W^4(r(5,1,2))$ by (†), and $W^4(r(5,1,2)) \approx W^4(r(5,2,3))$ by (‡). So, (4.3) now follows from (4.2). The Kirby diagram for $W^4(r(a,0,1)$ is a Hopf link with framings a and 0; by [GS99, pp. 127, 130, & 144], the corresponding 4-manifold may be closed up with a 4-handle yielding $S^2 \times S^2$ if a is even and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ if a is odd. This proves (4.4). Lastly, (‡) gives $W^4(r(c+1, c-1, c)) \approx W^4(r(c+1, 0, 1))$, and (4.5) now follows from (4.4). $\square$

Corollary 4.3. *The closed 4-manifolds appearing as $X^4(r)$ for an Artin presentation r on n-generators for $n = 0$, 1, and 2 are exactly: S^4 for $n = 0$, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ for $n = 1$, and $\mathbb{C}P^2 \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $\overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$, and $S^2 \times S^2$ for $n = 2$.*

Remark 4.4. As noted by a referee, an alternative proof of Corollary 4.3 may be obtained using Corollary 1.4 from Meier and Zupan [MZ17]. Their approach utilizes trisections of 4-manifolds and does not appear to provide alternative proofs of Theorems 3.3 and 4.2 herein.

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