Knot Theory and the Casson Invariant in Artin Presentation Theory

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1 Introduction

Artin presentation theory (AP theory) is the study of certain finite group presentations that are intimately related to smooth, compact, simply-connected 4-manifolds, closed, orientable 3-manifolds, and knots and links therein [W],[CW].

By definition, an Artin presentation, r, is a finite presentation:

$$r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

satisfying the following equation in F_n (the free group on $x_1, \ldots, x_n$):

$$x_1 x_2 \cdots x_n = \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right).$$

An Artin presentation, r, determines a smooth, compact, connected, simply-connected 4-manifold $W^4(r)$ with boundary $M^3(r)$ a closed, orientable, 3-manifold. The fundamental group of $M^3(r)$ is isomorphic to $\pi(r)$, the group presented by r, and all closed, orientable 3-manifolds appear in this way (see Section 2 ahead). Thus, Artin presentations characterize the fundamental groups of closed, orientable 3-manifolds (this result is due to F. González-Acuña [GA]).

An important theme in AP theory is that invariants of the 3- and 4-manifolds should be computed group theoretically solely in function of the discrete Artin presentation r. This was done for the Rohlin invariant of an integral homology 3-sphere $M^3(r)$ by González-Acuña [GA]. For simplicity, assume r is an Artin presentation whose exponent sum matrix $A(r)$ is the identity[1]. The associated presentation:

$$\langle x_1, \ldots, x_n \mid x_1 r_1 = r_1 x_2, x_2 r_2 = r_2 x_3, \ldots, x_{n-1} r_{n-1} = r_{n-1} x_n \rangle$$

abelianizes to $\mathbb{Z}$ and so has an Alexander polynomial Δ. Let $d = \Delta(-1)$. Then:

$$\mu\left(M^3(r)\right) = \frac{d^2-1}{8} \bmod 2.$$

[1] It is open whether every integral homology 3-sphere is represented by an Artin presentation with $A(r) = I$. The analogue is true for Heegaard decompositions.

The main purpose of this paper is to give such a formula for the Casson invariant of any rational homology 3-sphere. Again, for simplicity assume r is an Artin presentation with $A(r) = I$. For $i = 1, \ldots, n$ let H_i be the associated presentation:

$$H_i = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{i-1}, x_{i+1}, \ldots, x_n \rangle .$$

Let Δ_i denote the Conway normalized Alexander polynomial of the group presented by H_i (i.e. $\Delta_i(1) = 1$ and $\Delta_i(t) = \Delta_i(t^{-1})$). Then:

$$\lambda\left(M^3(r)\right) = \frac{1}{2}\sum_{i=1}^{n} \Delta_i''(1).$$

Recall that Δ_i can be computed group theoretically in function of H_i (which is in function of r) using the Fox free calculus and a computer algebra system such as MAGMA. The general formula where $M^3(r)$ is a rational homology 3-sphere is in Section 3 below.

Remark: simple examples show that λ is not merely the integer $\left(d^2 - 1\right)/8$ appearing in González-Acuña's formula above.

This shows that all 3D Seiberg-Witten invariants can be computed group theoretically in AP theory by a result of Lim [Lim]. An open problem is to find an analogue of Floer theory in AP theory. The reader is reminded that the Casson invariant is the Euler characteristic of the Floer homology groups.

It is not known which 4-manifolds appear in AP theory. For example, it is not known if every smooth, compact, connected, simply-connected 4-manifold with a connected and simply-connected boundary is a $W^4(r)$, see [GS], p.344, for a related problem. Many interesting 4-manifolds are known to appear, such as all elliptic surfaces $E(n)$, in particular the Kummer surface $K3 = E(2)$ (see [CW]). Thus, there is a nontrivial theory of Donaldson and Seiberg-Witten invariants in AP theory. An interesting question is: how can one compute smooth invariants of the 4-manifold $W^4(r)$ solely in function of the discrete Artin presentation, r? It seems natural to think that the canonical knot theory of AP theory, which is intimately related to the Casson invariant formula, and the planarity of the page in the open book construction will be relevant to this question. Recall that relationships between Alexander polynomials of knots and smooth invariants have already surfaced [MT],[FS].

Remark: in certain special cases the Casson invariant was obtained in a different manner from surgery on closed pure 3-braids [AS].

2 Knots and Links

The fundamentals of AP theory appeared in [W], and also [CW] and [C]. Some of these results are recalled here for convenience.

$\mathcal{R}_n$ denotes the set of Artin presentations on n generators and forms a group canonically isomorphic to $P_n \times \mathbb{Z}^n$, where P_n is the classical pure braid group on n strands. In this way an Artin presentation $r \in \mathcal{R}_n$ determines $h(r)$, a self homeomorphism of the compact 2-disk with n holes, Ω_n, that is the identity on $\partial\Omega_n$ and is unique up to isotopy rel $\partial\Omega_n$. Applying the open book construction to $h(r)$ with planar page Ω_n one obtains $M^3(r)$, a closed, connected, orientable 3-manifold. The fundamental group of $M^3(r)$ is isomorphic to $\pi(r)$, the group presented by r.

The Artin presentation r also determines $W^4(r)$ a smooth, compact, connected, simply-connected 4-manifold with boundary $\partial W^4(r)$ equal to $M^3(r)$. $W^4(r)$ can be constructed in two equivalent ways, either by a generalization of the open book construction [W] or by adding 2-handles to the 4-disk according to the closure of the framed pure braid determined by r [CW].

Let $A(r)$ denote the exponent sum matrix of r, i.e. $[A(r)]_{ij}$ = exponent of x_i in abelianized r_j. $A(r)$ is an $n \times n$ integer matrix that presents $H_1(M^3(r);\mathbb{Z})$. For an Artin presentation r, $A(r)$ is symmetric and represents the quadratic form of $W^4(r)$. So, the integer (co)homology of both $M^3(r)$ and $W^4(r)$ is given simply by $A(r)$. In particular, $M^3(r)$ is a rational homology 3-sphere if and only if $\det A(r) \neq 0$ and is an integral homology 3-sphere if and only if $\det A(r) = \pm 1$.

The knot and link theory in AP theory is at once canonical and sufficiently general. Let $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ be an Artin presentation in $\mathcal{R}_n$. The boundary of the planar page Ω_n in the open book construction of $M^3(r)$ determines $n+1$ distinguished knots $k_0, k_1, \ldots, k_n$ in $M^3(r)$. The knot groups G_i of the knots $k_i = k_i(r)$ are presented by ([W], p.226,227 and [CW], Section 2.1):

$$\begin{aligned} G_0 &= \langle x_1, \ldots, x_n \mid r_1 = r_2 = \cdots = r_n \rangle, \\ G_i &= \langle x_1, \ldots, x_n \mid r_1, r_2, \ldots r_{i-1}, r_{i+1}, \ldots, r_n \rangle, \ i \neq 0. \end{aligned}$$

Although this is not used here, if $M^3(r)$ is an integral homology 3-sphere then the peripheral structures of these knots are determined by $A(r)^{-1}$ [W].

The following is an unpublished result of González-Acuña and shows that the AP theory knot theory is sufficiently general. By taking L to be the empty link, one recovers González-Acuña's result [GA] that every closed, connected, orientable 3-manifold is homeomorphic to $M^3(r)$ for some Artin presentation r.

Theorem. *Let L be a link in a closed, connected, orientable 3-manifold M^3. Then, (M^3, L) is homeomorphic to $(M^3(r), K)$ for some Artin presentation r where K is the sublink $k_1, \ldots, k_m$ of the boundary of Ω_n.*

Proof. Let $l_1, \ldots, l_m$ be the components of L. Let Y be the subset of M^3 obtained from a tubular neighborhood $T(L)$ of L by connecting each component of $T(L)$ to a disjoint 3-disk $D^3 \subset M^3 - T(L)$ with an embedded 1-handle. Y is homeomorphic to the standard, orientable handlebody H_m of genus m. By attaching finitely more 1-handles to D^3 in M^3 (disjoint from $Y - D^3$) one obtains $Z \subset M^3$ such that Z is homeomorphic to H_g, $g \geq m$, and also $W = M^3 - intZ$ is homeomorphic to another copy H'_g of the standard handlebody; this follows from Morse/handle theory [GS], Chapter 4.

Following Lickorish [L1],[L2], the homeotopy group of ∂H_g is generated by Dehn twists about the simple curves $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, and $c_1, \ldots, c_{g-1}$ in

∂H_g where the $a_i s$ are not contractible in H_g. Then, Z is homeomorphic to the standardly embedded H_g in $\mathbb{R}^3$ such that l_i is parallel to a_i by the construction above. Moreover, M^3 is homeomorphic $H_g \cup_f H_g'$ for some homeomorphism f that is isotopic to a product of finitely many Dehn twists $De_1, \ldots, De_k$, where each e_i is one of the curves $a_1, \ldots, a_g$,$b_1, \ldots, b_g$,$c_1, \ldots, c_{g-1}$ (see [L2]). One may assume that $e_i = a_i$ for $i = 1, 2, \ldots, m$, since if one performs the Dehn twists $Da_1, \ldots, Da_m$, $Da_1^{-1}, \ldots, Da_m^{-1}$, and then $De_1, \ldots, De_k$, the resulting homeomorphism of ∂H_g is isotopic to f.

As Lickorish showed [L1] each Dehn twist Dx can be accomplished by performing ± 1 surgery on a knot in the interior of H_g that is parallel x. Let s_i be a knot in the interior of H_g that is parallel to a_i for $i = 1, \ldots, m$ such that l_i is a longitude of s_i that does not link s_i. Since s_i has framing ± 1, one can slide l_i over s_i by isotopy so that l_i becomes a meridian of s_i. Each of the remaining Dehn twists contributes a knot to be surgered; these are all disjoint and each is disjoint from a neighborhood of each s_i that contains l_i as a meridian. Let β be the union of all the knots to be surgered (including the s_i). It follows from [L3], p.418-419, or [R], p.279,340,341, that β is isotopic to the closure of a pure braid. The result follows since each component l_i of the link L is a meridian of a component of β. ■

There are other useful knot groups associated to an Artin presentation.

Fix $r \in \mathcal{R}_n$. Let $\beta = \beta(r)$ denote the framed pure braid determined by r. For $i = 1, \ldots, n$ let β_i denote the ith component of β with framing $a_i = [A(r)]_{ii}$. Let $M^3(\beta_1, \ldots, \beta_k)$ denote the closed, orientable 3-manifold obtained by just performing surgery on the closure of the first k components of β. Notice that by performing j-reduction (see [W] or [C]) on r for $j = k+1, k+2, \ldots, n$ one obtains an Artin presentation s such that $M^3(s)$ is homeomorphic to $M^3(\beta_1, \ldots, \beta_k)$. Notice further that the closure of β_k is a knot in $M^3(\beta_1, \ldots, \beta_{k-1})$ whose knot group is presented by:

$$H_i = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{i-1}, x_{i+1}, \ldots, x_n \rangle .$$

This follows from the HNN construction (see [W], p.247).

The knot groups G_i and H_i will both be utilized in the following section.

3 The Casson Invariant

The formula described here for the Casson invariant of a rational homology 3-sphere in AP theory is simplest in case $A(r)$ is a unimodular diagonal matrix.

Let $r \in \mathcal{R}_n$ be an Artin presentation such that $A(r)$ is a diagonal matrix with $[A(r)]_{ii} = \epsilon_i = \pm 1$. Notice that $M^3(r)$ is an integral homology 3-sphere. For $i = 1, \ldots, n$ let Δ_i denote the Conway normalized Alexander polynomial of the presentation H_i described above.

Theorem. $\lambda\left(M^3(r)\right) = \frac{1}{2}\sum_{i=1}^{n} \epsilon_i \Delta_i''(1)$.

This follows from the discussion at the end of the previous section and [AMc]. The general case where $\det A(r) \neq 0$ is similar but more technical.

Notice that $\Delta_1 = 1$ always (each component of a closed pure braid is an unknot), and so the sum need only be taken over $i = 2, \ldots, n$.

Definition: *let A be an $n \times n$ integer matrix. Let $A_{1\cdots k}$ denote the upper left $k \times k$ minor of A. Then A is* permissible *provided* $\det A_{1\cdots k} \neq 0$ *for* $k = 1, \ldots, n$.

Let $r \in \mathcal{R}_n$ be an Artin presentation and $1 \leq k \leq n$, then $A(r)_{1\cdots k}$ is a presentation matrix of $H_1\left(M^3(\beta_1, \ldots, \beta_k); \mathbb{Z}\right)$. If $A(r)$ is permissible then $M^3(\beta_1, \ldots, \beta_k)$ is a rational homology 3-sphere for $k = 1, \ldots, n$ (in particular, $M^3(r)$ is a rational homology 3-sphere). So, one can obtain $M^3(r)$ by a sequence of surgeries on $\beta_1, \ldots, \beta_n$ and at each stage one will have a rational homology 3-sphere. This shows that the notion of permissible here agrees with that in [Wa], p.96.

Theorem. *If $r \in R_n$ has $A(r)$ permissible then:*

$$\lambda\left(M^3(r)\right) = w_0 + \sum_{i=1}^{n} w_i \Delta_i''(1)$$

for rational weights w_j that are determined by $A(r)$.

This follows from the discussion above and [Wa], p.95,96. Notice that the homological data in Walker's formula is strictly in function of $A(r)$ and a computer can be programmed to compute these numbers.

It remains to determine the Casson invariant of an Artin presentation $r \in \mathcal{R}_n$ where $\det A(r) \neq 0$ and $A(r)$ not permissible. A procedure for reducing to the permissible case is described in [Wa], p.105,106, although there are drawbacks to his method. First, it destroys the pure braid thus removing one from the realm of AP theory. Second, it is unduly complicated and nonconstructive. Below is a simple and constructive method of reducing to the permissible case.

Let D denote a diagonal $n \times n$ matrix where each diagonal entry $d_i \in \{-1, 0, 1\}$.

Claim. *Suppose A is an $n \times n$ integer matrix with* $\det A \neq 0$*. Then, for some choice of D the matrix $A + D$ is permissible and* $\det(A + D_{1\cdots k}) \neq 0$ *for* $k = 1, \ldots, n$.

Proof. If A is permissible, then let $D = 0$. Otherwise, here is a constructive way to choose D. Choose d_1 so that $\det(A_{1\cdots 1} + D_{1\cdots 1}) \neq 0$ and $\det(A + D_{1\cdots 1}) \neq 0$. Expanding these determinants gives two linear equations in d_1 that one wishes to not vanish (the first with nonzero nonconstant coefficient and the second with nonzero constant coefficient), and with three possible values for d_1 such a choice is always possible.

Having chosen $d_1, \ldots, d_{k-1}$, choose d_k so that $\det(A_{1\cdots k} + D_{1\cdots k}) \neq 0$ and $\det(A + D_{1\cdots k}) \neq 0$. Again, expanding these two determinants gives two linear equations in d_k that one wishes to not vanish (the first with nonzero nonconstant coefficient and the second with nonzero constant coefficient) and so such a choice is always possible. Having chosen D in this way, the result follows. ∎

Finally, let $r \in \mathcal{R}_n$ be an Artin presentation such that $\det A(r) \neq 0$ and $A(r)$ is not permissible. Let D be a matrix given by the claim applied to $A(r)$. $M^3(r)$ has a surgery diagram given by closure of the pure braid β. For each $i = 1, \ldots, n$, if $d_i \neq 0$ then introduce a meridian to β_i with framing ∞ in the surgery diagram of $M^3(r)$. This does not change the 3-manifold; notice that the meridian to β_i is in fact $k_i = k_i(r)$ (an important point!). Now, perform a Rolfsen twist ([R], p.264-267 or see [GS], p.162,163) in the correct direction (+ or - depending on d_i) that simply changes framings in the diagram: the framing $a_i = [A(r)]_{ii}$ of β_i becomes $a_i + d_i$ and the framing ∞ of k_i becomes d_i. One obtains $M^3(r)$ by surgering (in this order) $\beta_1, \ldots, \beta_n$ with new framings and then $k_n, \ldots, k_1$ with framings d_i; here one skips any k_i where $d_i = 0$. Then, by the choice of D from the claim, one has a rational homology 3-sphere at each stage. Moreover, all of the knot groups in this series of successive surgeries is either a G_i or an H_i and so is determined by r. The Casson invariant of $M^3(r)$ is now computed using the Alexander polynomials of the presentations G_i and H_i similar to the permissible case and all of the required homological data is given by $A(r)$ and D.

References

[AS] Armas-Sanabria, L., *The μ invariant and the Casson invariant of 3-manifolds obtained by surgery on closed pure 3-braids*, JKTR **13**, No.3, 2004, 427-440.

[AMc] Akbulut, S. and McCarthy, J., *Casson's invariant for oriented homology 3-spheres*, Math. Notes **36**, Princeton University Press, 1990.

[C] Calcut, J. S., *Torelli actions and smooth structures on 4-manifolds*, Ph.D. Thesis, University of Maryland, 2004.

[CW] Calcut, J. S. and Winkelnkemper, H.E., *Artin presentations of complex surfaces*, Bol. Soc. Mat. Mexicana (3), Vol. 10, Special Issue in honor of F. González-Acuña, 2004, 63-87.

[FS] Fintushel, R. and Stern, R., *Knots, links, and 4-manifolds,* Invent. Math. **134**, No.2, 1998, 363-400.

[GA] González-Acuña, F., *Open books*, Lecture notes, University of Iowa, 1975.

[GS] Gompf, R. and Stipsicz, A., *4-manifolds and Kirby calculus,* GSM vol.20, AMS, 1999.

[L1] Lickorish, W., *A representation of orientable combinatorial 3-manifolds*, Ann. Math. **76**, no.3, Nov. 1962, 531-540.

[L2] Lickorish, W., *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Camb. Phil. Soc. **60**, 1964, 769-778.

[L3] Lickorish, W., *A foliation for 3-manifolds*, Ann. Math. **82**, 1965, 414-420.

[Lim] Lim, Y., *The equivalence of Seiberg-Witten and Casson invariants for homology 3-spheres*, Math. Res. Lett., **6**, 1999, 631-643.

[MT] Meng, G. and Taubes, C.H., $\underline{SW}$ *=Milnor torsion,* Math. Res. Lett. **3**, No.5, 1996, 661-674.

[R] Rolfsen, D., *Knots and links*, Publish or Perish, Berkeley, 1976.

[Wa] Walker, K., *An extension of Casson's invariant*, Ann. Math. Studies **126**, Princeton University Press, 1992.

[W] Winkelnkemper, H.E., *Artin presentations I: Gauge theory, 3+1 TQFTs and the braid groups,* JKTR, vol.11, No.2, 2002, 223-275.