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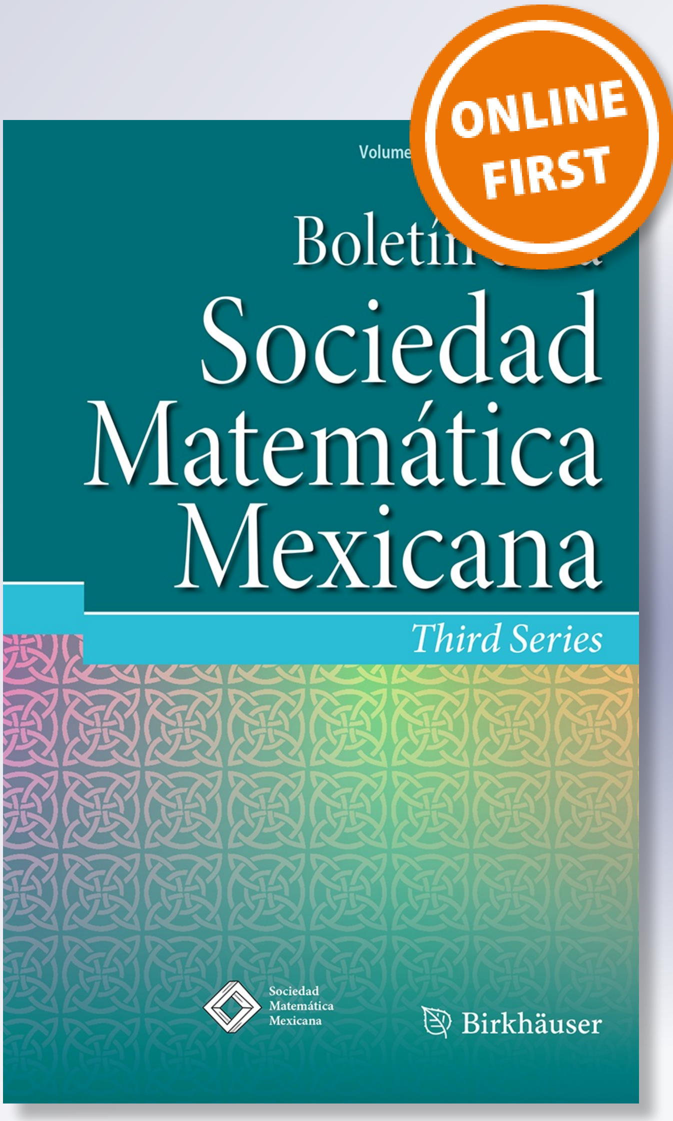

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ORIGINAL ARTICLE

The explicit algebraic autonomy of Artin presentation theory and the Fox Calculus. I

J. S. Calcut[1] · **H. E. Winkelnkemper**[2]

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Abstract Given an Artin presentation r, we use the Fox Calculus to obtain a short, purely algebraic computation, just in function of r, of the second homotopy group of the associated manifold $M^3(r)$. This allows us to give a Langlands-like formulation (not yet a proof) of the three-dimensional Borel conjecture for closed, orientable 3-manifolds, the latter being a theorem of geometrization theory and implying the Poincaré conjecture.

Keywords Artin presentation · 3-Manifold · Second homotopy group · Fox Calculus

Mathematics Subject Classification Primary 57N10; Secondary 57M05 · 57M50

1 Introduction

Artin Presentation theory (AP theory [3,13]) is a purely discrete, group-theoretic (in fact, pure framed braid-theoretic) theory which *encodes* a large class of compact, connected, simply-connected, *smooth* 4-manifolds with a connected boundary. Given an Artin presentation r, such a 4-manifold $W^4(r)$ and its connected boundary $M^3(r)$ are topologically constructed *in unison* with an open book construction in a non-

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analytic, non-infinitesimal, but smooth manner, so that r presents the fundamental group, $\pi(r) = \pi_1\left(M^3(r)\right)$, of $M^3(r)$ and the exponent sum matrix, $A(r)$, which is always symmetric represents the quadratic form of $W^4(r)$.

All symmetric integer $n \times n$ matrices appear in this manner and all connected, closed, orientable 3-manifolds can be so represented.

In particular, all closed, orientable 3-manifolds come provided, *ab initio*, with such an intimately related ('at birth', so to speak) smooth, compact, simply-connected four-dimensional cobordism and its arbitrary omission, from the beginning in low-dimensional topology, could, in our opinion, make classical M^3-theory 'incomplete' in some sense, despite the success of the Geometrization Program. We quote Thurston himself [12, p. 177]: "Even if a theorem about Haken manifolds can be proven using geometric techniques, there is high value in finding purely topological techniques to prove it."

From Donaldson's analytic, differential geometric Yang–Mills theory using theorem, one immediately obtains the following theorem, which, nevertheless, *can be completely stated in AP theory in a purely algebraic, non-analytic fashion*.

Theorem 1 (Winkelnkemper [13, Thm. I, p. 240]) *If $A(r)$ is a unimodular, $n \times n$ symmetric integer matrix, which by Donaldson's theorem cannot represent the quadratic form of a closed, smooth, simply-connected 4-manifold (e.g., a matrix such as E_8), then the group $\pi(r)$ cannot be trivial. In fact, $\pi(r)$ has a non-trivial representation into $SU(2)$.*

Thus, in particular, in AP Theory, Donaldson's Theorem singles out and isolates $SU(2)$ from all other compact Lie groups.

Although Theorem 1 already relates the two most important theories (geometrization and $(3+1)$-dimensional Yang–Mills theory) of low-dimensional topology, this purely algebraic theorem still depends *unnaturally* for its proof on the analytic PDE using analysis of Donaldson's YM Theory. This is the first of a series of papers attempting to unite Thurston and Donaldson theories in a sharper manner by aiming to provide a purely discrete, group-theoretic proof, an AP theoretic proof, of the above theorem. We would consider this a first important meta-mathematical step in uniting these two most important theories. Since the matrix $A(r)$ is the 'Alexander matrix' of the presentation r (see Crowell and Fox [4, p. 100]), the Fox Calculus seems to be the natural arena for solving this problem. This problem is also relevant in a more general sense in the theory of PDEs as described by Klainerman [8].

Our natural strategy is, given an Artin presentation r, to find purely algebraically, *in function of r only*, all of the important topological invariants of $W^4(r)$ and $M^3(r)$. This has already been done, for example, for the Casson invariant [2,6], but in this paper we go deeper and determine the *second* homotopy group $\pi_2(r)$ of $M^3(r)$, in particular determining in function of r only, when $M^3(r)$ is irreducible and/or aspherical, *very important conditions in Geometrization theory* (see also Remark 2 below).

In this paper, we show how the Artin equation allows the Fox Free Calculus to completely determine $\pi_2(r)$ in a purely algebraic, discrete, group-theoretic manner: assume r is an Artin presentation on n generators, $x_1, x_2, \ldots, x_n$. Let $\mathbb{Z}\pi(r)$ be the integral group ring of the group $\pi(r)$. Let $\mathrm{Nul}\,(r)$ be the set of horizontal n-vectors v in $\mathbb{Z}\pi(r)$ such that $v \operatorname{Jac} r = 0$ in $\mathbb{Z}\pi(r)$, where $\operatorname{Jac} r$ denotes the Jacobian of r as in

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the Fox Calculus. By simply deriving the Artin equation with respect to each x_i we obtain the following preliminary lemma.

Lemma 1 *The vector*

$$V(r) = \big(x_1 - 1, x_1(x_2 - 1), x_1x_2(x_3 - 1), \ldots, x_1x_2 \cdots x_{n-1}(x_n - 1)\big)$$

of $\mathbb{Z}\pi(r)$ always *lies in Nul* (r).

Theorem 2 $\pi_2(r) = \mathit{Nul}\,(r) \,/\, \langle V(r)\rangle$*, where* $\langle V(r)\rangle$ *is generated by* $V(r)$ *in* $\mathbb{Z}\pi(r)$.

Remark 1 This actually is an isomorphism of $\mathbb{Z}\pi(r)$ modules as in [10,11].

Remark 2 This raises the question whether there exists an *algebraic* algorithm for deciding whether or not Nul (r) is linear.

Thus, $\pi_2(r) = 0$ if and only if Nul (r) is a 'line' in $\mathbb{Z}\pi(r)$. To explain why *linearity* should be related to *irreducibility*, we interpret the three-dimensional Borel conjecture for closed, orientable 3-manifolds [1, p. 17] as a statement analogous to that of the Geometric Langlands Program (see Frenkel [5]):

For Artin presentations r such that $\pi(r)$ is not finite cyclic and Nul (r) is linear, there is a bijection between the $\pi(r)$ (the left algebraic side of the Langlands correspondence) and the $M^3(r)$ (the right analytic, topological, automorphic side).

Thus, we have a bijection, just like in the Geometric Langlands correspondence, involving Riemann surfaces, but instead of *flat* bundles over a Riemann surface, we have *linear* Artin presentations, i.e., r where Nul (r) is *linear*; and instead of automorphic L-packets [5, p. 6] we have irreducible, connected, closed, orientable 3-manifolds. This analogy with automorphy is even stronger when $\pi(r)$ is infinite, but not $\mathbb{Z}$, i.e., when the $M^3(r)$ are aspherical, because then the universal cover is homeomorphic to $\mathbb{R}^3$.

Our proof of Theorem 2 is based on a recent theorem of Lei and Wu [9, Theorem 1.1]. Their theorem is algebraic and topological in nature making no essential use of Geometrization.

We thank M. A. Gutierrez and J. A. Schafer for very helpful conversations.

2 Proofs

Theorem 2 will follow from Lei and Wu [9, Theorem 1.1] due to three fortuitous facts from AP theory (discussed further below).

1. The simplified structure of the Heegaard homeomorphism in AP theory (see Winkelnkemper [13, Figure 1.3]). In short, it is the identity except on the bottom half of the standard genus n surface Σ_n.
2. A very convenient new set of generators of $\pi_1(\Sigma_n)$, in place of the classical ones used by Lei and Wu, as defined by Winkelnkemper [13, Figure 1.3].
3. The computation described in our preliminary Lemma 1 which permits us to state Theorem 2 in such a succinct, conceptually simple manner with 'linear algebra' in the group ring. This is a consequence of the simple form assumed by the Artin equation with respect to the Fox Calculus.

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Fix an Artin presentation r on n generators $x_1, x_2, \ldots, x_n$. This means that:

$$r = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$$

is a balanced, finite presentation such that the following holds in the rank n free group F_n:

$$x_1 x_2 \cdots x_n = r_1^{-1} x_1 r_1 r_2^{-1} x_2 r_2 \cdots r_n^{-1} x_n r_n. \tag{1}$$

Equation (1) is the fundamental *Artin equation*. Fundamental groups of closed, oriented 3-manifolds are exactly those groups that admit Artin presentations by González-Acuña [6].

The following notation will be used:

$$\begin{aligned} A^B &:= B^{-1} A B, \\ (A, B) &:= A^{-1} B^{-1} A B, \\ [A, B] &:= A B A^{-1} B^{-1} = \left(A^{-1}, B^{-1}\right). \end{aligned}$$

The first two definitions are consistent with the algebraic computation engine MAGMA and the third agrees with Lei and Wu [9, p. 894]. In particular, the Artin equation is:

$$x_1 x_2 \cdots x_n = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}.$$

Let $\Sigma_n \subset \mathbb{R}^3$ denote the standard closed, oriented surface of genus n that is the boundary of the standard handlebody $H_n \subset \mathbb{R}^3$ (see Fig. 1).

The AP theory generators for $\pi_1(\Sigma_n)$ were defined by Winkelnkemper [13, p. 249]. By reflecting Σ_n across a horizontal plane, these generators appear as in Fig. 1. We have:

$$\pi_1(\Sigma_n) = \left\langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \mid x_1 x_2 \cdots x_n = x_1^{y_1} x_2^{y_2} \cdots x_n^{y_n} \right\rangle.$$

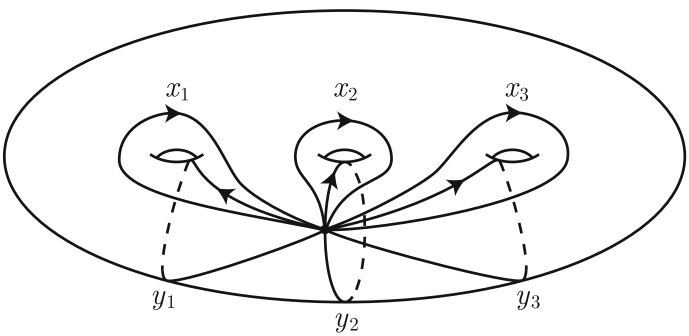

Fig. 1 AP theory generators for $\pi_1(\Sigma_3)$

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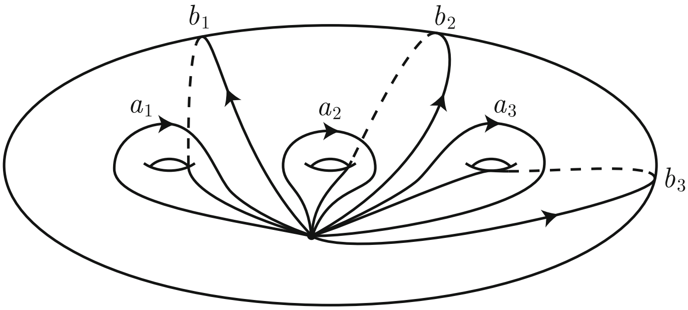

Fig. 2 Classical generators for $\pi_1(\Sigma_3)$

Classical generators for $\pi_1(\Sigma_n)$, as in Fig. 2, yield:

$$\pi_1(\Sigma_n) = \langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \mid [a_1, b_1][a_2, b_2] \cdots [a_n, b_n] = 1\rangle.$$

Notice that $a_i = x_i$ for each $1 \leq i \leq n$, and $\pi_1(H_n) = \langle a_1, a_2, \ldots, a_n\rangle$ in agreement with [9, p. 888].

The curves $x_i = a_i$, y_i, and b_i, $1 \leq i \leq n$, may be simultaneously drawn on Σ_n such that any two of them meet only at the common basepoint. To express the classical generators in terms of the AP theory generators, cut Σ_n along the x_is and the y_is. This yields a compact 2-disk, D, whose boundary consists of segments labelled by the symbols $x_i^{\pm 1}$ and $y_i^{\pm 1}$ for $1 \leq i \leq n$. The b_is are disjoint chords in D, each of which may be isotoped (relative to endpoints) into ∂D in essentially two ways. Similarly, one may express the AP theory generators in terms of the classical generators. Below are the resulting formulae.

Consider the Artin presentation:

$$r = \langle a_1, a_2, \ldots, a_n \mid r_1, r_2, \ldots, r_n\rangle$$

and a new basis $C = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ for F_{2n} given by:

$$\begin{aligned} u_i &= a_i \text{ for } 1 \leq i \leq n, \\ v_i &= b_i^{-1} \prod_{k=i}^{1} [b_k, a_k] \text{ for } 1 \leq i \leq n-1, \text{ and} \\ v_n &= b_n^{-1}. \end{aligned}$$

The u_i and v_i are as in Figure 1.3, p. 249, of [13] with $u_i = x_i$ and $v_i = y_i$.

For $n = 1$, we have:

$$\begin{aligned} b_1 &= v_1^{-1}, \\ v_1 &= b_1^{-1}. \end{aligned}$$

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For $n = 2$, we have:

$$\begin{aligned}
b_1 &= (u_1)^{-1}\left(v_1^{-1}u_1\right) = u_2\left(v_1 u_2^{v_2}\right)^{-1},\\
b_2 &= (u_1u_2)^{-1}\,u_1^{v_1}\left(v_2^{-1}u_2\right) = (v_2)^{-1},\\
v_1 &= b_1^{-1}\,[b_1, a_1] = b_1^{-1}\,[a_2, b_2],\\
v_2 &= b_2^{-1}\,[b_2, a_2]\,[b_1, a_1] = b_2^{-1}.
\end{aligned}$$

For $n = 3$, we have:

$$\begin{aligned}
b_1 &= (u_1)^{-1}\left(v_1^{-1}u_1\right) = u_2u_3\left(v_1 u_2^{v_2}u_3^{v_3}\right)^{-1},\\
b_2 &= (u_1u_2)^{-1}\,u_1^{v_1}\left(v_2^{-1}u_2\right) = u_3\left(v_2 u_3^{v_3}\right)^{-1},\\
b_3 &= (u_1u_2u_3)^{-1}\,u_1^{v_1}u_2^{v_2}\left(v_3^{-1}u_3\right) = (v_3)^{-1},\\
v_1 &= b_1^{-1}\,[b_1, a_1] = b_1^{-1}\,[a_2, b_2]\,[a_3, b_3],\\
v_2 &= b_2^{-1}\,[b_2, a_2]\,[b_1, a_1] = b_2^{-1}\,[a_3, b_3],\\
v_3 &= b_3^{-1}\,[b_3, a_3]\,[b_2, a_2]\,[b_1, a_1] = b_3^{-1}.
\end{aligned}$$

The obvious pattern continues for larger n.
Define ϕ by its action on C:

$$\begin{aligned}
\phi\,(u_i) &= r_i^{-1}u_ir_i,\\
\phi\,(v_i) &= r_i^{-1}v_i.
\end{aligned}\tag{2}$$

where the r_i are written as words in the $u_i = a_i$ and $1 \le i \le n$. This is a well-defined automorphism and the fundamental group G of the resulting manifold M is presented by r. Next, we recall Lei and Wu's [9, Theorem 1.1]. Suppose T is an endomorphism of $(\mathbb{Z}G)^n$ with matrix $\left\| D_j\,(\phi\,(b_i))\right\|$ in the standard basis. Further, suppose that Λ is the submodule of $(\mathbb{Z}G)^n$ generated by $(a_1 - 1, \ldots, a_n - 1)^t$. Then, M is aspherical if and only if the following sequence is exact:

$$0 \to \Lambda \to (\mathbb{Z}G)^n \xrightarrow{T} (\mathbb{Z}G)^n \xrightarrow{\theta} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Here, $\theta : (\gamma_1, \ldots, \gamma_n)^t \mapsto \sum \gamma_i^{-1}$ and ϵ is the augmentation homomorphism. We must translate Lei and Wu's theorem into a result using the basis C. First, IF_{2n} is free with basis $\left\{a_i - 1, b_j - 1\right\}$ (or $\left\{u_i - 1, v_j - 1\right\}$). If $I'F_{2n}$ denotes IF_{2n} itself but equipped with multiplication $w \cdot (a - 1) := \phi^{-1}\,(w)\,(a - 1)$ where $w, a \in F$, then $I'F$ is isomorphic to IF and the mapping $(a - 1) \mapsto (\phi\,(a) - 1)$ is F-linear. Thus, a matrix $K = \left\| \frac{\partial u, v}{\partial a, b} \right\|$ expresses the change of basis transformation from B to C. If T_B and T_C are the matrices of T with bases B and C, respectively, then we have:

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$$T_C = K T_B K^{-1}. \tag{3}$$

We will study K by looking at the four obvious submatrices. The upper left corner is $\left\| \frac{\partial u_i}{\partial a_j} \right\|$ which equals the $n \times n$ identity matrix. Similarly, the upper right corner is $\left\| \frac{\partial u_i}{\partial b_j} \right\|$ which equals the $n \times n$ zero matrix. Let $X = \left\| \frac{\partial v_i}{\partial a_j} \right\|$ and $Z = \left\| \frac{\partial v_i}{\partial b_j} \right\|$ denote the remaining lower left and lower right corners, respectively.

Lemma 2 *The image under q of X, denoted X^q, is the $n \times n$ zero matrix.*

Proof We have $D_j a_i b_i^{-1} a_i^{-1} = (1 - a_i b_i^{-1} a_i^{-1})\delta_{ij}$ and $D_j\,[b_k, a_k] = (b_k - [b_k, a_k])$ $\delta_j k$. As $\frac{\partial v_i}{\partial a_j}$ is a linear combination of these two expressions, its image under q is zero. $\square$

As K is invertible, Z must be invertible. Thus, Z^q is invertible as well. We have $Z^q = \left\| q\left(\frac{\partial v_i}{\partial b_j}\right) \right\|$ and $(Z^q)^{-1} = \left\| q\left(\frac{\partial b_i}{\partial v_j}\right) \right\|$. The four corners of K^{-1} are, counterclockwise from the upper left corner, the identity, $-Z^{-1}X$, Z^{-1}, and the zero matrix. In particular, the four corners of $(K^q)^{-1}$ are the identity, the zero matrix, $(Z^q)^{-1}$, and the zero matrix. In (3), we are interested only in the $\left\| \frac{\partial \phi(a_j)}{\partial b_i} \right\|$ corner (the lower left corner) and it is the matrix sum of $Z \left\| \frac{\partial \phi(b_i)}{\partial a_j} \right\|$ and several other matrices either premultiplied by X or postmultiplied by $-Z^{-1}X$ (or both). Upon composing with q, we obtain the following.

Lemma 3 *The matrix T_C of T with respect to the basis C is the product ZT_B.*

Now, define $p : F_{2n} \to G$ to be the composition of q and the quotient map $F_n \to G$. Let $S = Z^p$, an invertible matrix obtained from Z by killing all coefficients b_j and all words in the a_i that lie in the normal closure of the r_i. This is really the matrix of the linear transformation T.

Corollary 4 *The matrix of the linear transformation T is the product ST when viewed as a matrix in the basis C.*

To compute S, we use (2). Cancelling the b_j and working modulo the r_i, we obtain:

$$\frac{\partial v_i}{b_j} = -\left(\delta_{ij} a_i + \sum_{k=1}^{i-1} (a_k - 1)\, \delta_{kj} \right).$$

Thus, we obtain the following.

Proposition 5 *The matrix S is the $n \times n$ lower triangular matrix:*

$$S = - \begin{bmatrix} a_1 & & & & & \\ a_1 - 1 & a_2 & & & & \\ a_1 - 1 & a_2 - 1 & a_3 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 - 1 & a_2 - 1 & a_3 - 1 & \cdots & a_{n-1} - 1 & a_n \end{bmatrix}.$$

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The diagonal entries of S are units. We proceed to compute the inverse of S. We use the method of reducing $[S \mid I]$ into $\left[I \mid S^{-1}\right]$ by row operations. The result of this tedious computation is as follows. For $1 \leq i, j \leq n$, define ω'_{ij} by:

$$\omega'_{ij} = \begin{cases} 0 & \text{for } i < j \\ 1 & \text{for } i = j \\ \left(a_i^{-1} - 1\right)\omega'_{i-1,j} + \omega'_{i-1,j} & \text{for } i > j. \end{cases}$$

Define $\omega_{ij} = a_i^{-1}\omega'_{ij}$ and we have the following.

Proposition 6 *The matrix* $S^{-1} = -\left\|\omega_{ij}\right\|$.

Recall that $T_C = \left\|\frac{\partial\phi(v_i)}{\partial u_j}\right\|$ and $\phi\left(v_i\right) = r_i^{-1}v_i$. So, applying p we get $T_C = -\left\|\frac{\partial r_i}{\partial a_j}\right\|$ (we use the fact that $u_i = a_i$). That is, $T_C = -\mathrm{Jac}\, r$, the Jacobian matrix of the r_i.

Theorem 3 *The* 3*-manifold* M *is aspherical if and only if the following sequence is exact:*

$$0 \to \Lambda \to (\mathbb{Z}G)^n \xrightarrow{T} (\mathbb{Z}G)^n \xrightarrow{\theta} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

It remains to compute ΛS^{-1} and this may be done in two ways: either directly or by taking derivatives on both sides of the Artin equation. Either approach yields the following.

Proposition 7 *The one-dimensional submodule* ΛS^{-1} *is generated by:*

$$v = (a_1 - 1, a_1(a_2 - 1), a_1a_2(a_3 - 1), \ldots, a_1a_2 \cdots a_{n-1}(a_n - 1))^t\,.$$

Since the left term of (2) of Lei and Wu [9, pp. 889–890] is $\pi_2\,(M)$, our Theorem 2 follows.

3 Examples

This section presents a few examples.

1. Consider the Artin presentation $r = \langle x_1 \mid r_1\rangle$, where $r_1 = 1$. Then, $M^3\,(r) = S^1 \times S^2$, $\pi\,(r) = \mathbb{Z}$, and $\mathbb{Z}\pi\,(r)$ is the ring of Laurent polynomials in one variable, t, and with integer coefficients. As $\mathrm{Jac}\, r = [0]$, $\mathrm{Nul}\,(r)$ equals all of $\mathbb{Z}\pi\,(r)$. As $V\,(r) = (t-1)$, $\mathrm{Nul}\,(r)\,/\,\langle V(r)\rangle$ is indeed $\mathbb{Z} = \pi_2\,(r) = \pi_2\left(S^1 \times S^2\right)$. (Obviously, setting $t = 1$ in $\mathbb{Z}\pi\,(r)$ gives $\mathbb{Z}$.)
2. Consider the Artin presentation $r = \langle x_1, x_2 \mid r_1, r_2\rangle$, where

$$\begin{aligned} r_1 &= x_1^3(x_1x_2)^{-1}, \\ r_2 &= x_2^4(x_1x_2)^{-1}. \end{aligned}$$

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Then, $M^3(r) = L(5, 2)$ as in Winkelnkemper [13, p. 228]. Note that $\pi(r) = \mathbb{Z}_5$, where $x_1 = t$, $x_2 = t^2$,

$$\operatorname{Jac} r = \begin{bmatrix} t + t^2 & -t \\ -1 & 1 + t^2 + t^4 \end{bmatrix}$$

and $V(r) = \big(t - 1, t(t^2 - 1)\big)$. It is straightforward to verify that the conclusion of Lemma 1 holds in this example. Note that setting $t = 1$ in $\operatorname{Jac} r$ yields $A(r)$ as it always should.

3. Consider the Artin presentation $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$, where

$$\begin{aligned} r_1 &= x_1^3 (x_1 x_2)^{-2}, \\ r_2 &= x_2^5 (x_1 x_2)^{-2}. \end{aligned}$$

Then

$$\pi(r) = I(120) = \left\langle x_1, x_2 \mid x_1^3 = x_2^5 = (x_1 x_2)^2 \right\rangle.$$

Set $s = x_1$ and $t = x_2$. Then,

$$\operatorname{Jac} r = \begin{bmatrix} s + s^2 - st & -s - sts \\ -1 - st & 1 - s + t + t^2 + t^3 \end{bmatrix}$$

and $V(r) = \big(s - 1, s(t - 1)\big)$. Lemma 1 checks out. [Here, care must be taken because $I(120)$ is not abelian, so neither is the group ring $\mathbb{Z}I(120)$.]

4. Consider the Artin presentation $r = \langle x_1, x_2, x_3 \mid r_1, r_2, r_3 \rangle$, where

$$\begin{aligned} r_1 &= x_1^{-2} x_2 x_3 r_2, \\ r_2 &= x_2^{-1} x_3^{-1} x_2^{-1} x_1^{-1}, \\ r_3 &= x_3^2 x_2^{-1}. \end{aligned}$$

Then, according to González-Acuña (unpublished) $M^3(r) = L(13, 5)$ and $\pi(r) \cong \mathbb{Z}_{13}$ is generated by $x_3 = t$ (note that $x_1 = t^8$ and $x_2 = t^2$). We have

$$\operatorname{Jac} r = \begin{bmatrix} -1 - t^5 - t^{10} & -t^8 + t^{10} - t^{11} & -t^{10} + t^{12} \\ -1 & -t^8 - t^{11} & -t^{10} \\ 0 & -1 & 1 + t \end{bmatrix}$$

and $V(r) = \big(t^8 - 1, t^8(t^2 - 1), t^{10}(t - 1)\big)$. Again, Lemma 1 checks out.

5. Consider the Artin presentation $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$, where

$$\begin{aligned} r_1 &= x_1^2, \\ r_2 &= x_2^2. \end{aligned}$$

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Then, $M^3(r) = \mathbb{R}P^3 \# \mathbb{R}P^3$ and $\pi(r) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. We have (where $s = x_1$ and $t = x_2$)

$$\operatorname{Jac} r = \begin{bmatrix} 1+s & 0 \\ 0 & 1+t \end{bmatrix}$$

and $V(r) = \big(s-1, s(t-1)\big)$. Lemma 1 checks out. In this example, $V(r)$ does not generate $\operatorname{Nul}(r)$ in $\mathbb{Z}\pi(r)$. In particular, the vectors $(1-s, 0)$ and $(0, 1-t)$ both lie in $\operatorname{Nul}(r)$. Topological considerations yield the second of the following isomorphisms:

$$\operatorname{Nul}(r) / \langle V(r) \rangle \cong \pi_2\left(\mathbb{R}P^3 \# \mathbb{R}P^3\right) \cong \mathbb{Z}.$$

6. Consider the Artin presentation $r = \langle x_1, x_2 \mid r_1, r_2 \rangle$ where

$$r_1 = x_1^3,$$
$$r_2 = x_2^3.$$

Then, $M^3(r) = L(3,1) \# L(3,1)$ and $\pi(r) \cong \mathbb{Z}_3 * \mathbb{Z}_3$. We have (where $s = x_1$ and $t = x_2$)

$$\operatorname{Jac} r = \begin{bmatrix} 1+s+s^2 & 0 \\ 0 & 1+t+t^2 \end{bmatrix}$$

and $V(r) = \big(s-1, s(t-1)\big)$. Lemma 1 checks out. In this example, $V(r)$ does not generate $\operatorname{Nul}(r)$ in $\mathbb{Z}\pi(r)$. Topological considerations yield the second of the following isomorphisms:

$$\operatorname{Nul}(r) / \langle V(r) \rangle \cong \pi_2(L(3,1) \# L(3,1)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots .$$

Here, $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ is the infinite direct product of infinite cyclic groups.

4 Remarks and questions

We close with some remarks aiming to put Theorem 2 in its proper context.

1. As is well known, it is an important classical result of Hopf [7] that, given any finite complex K, the second homotopy group, $\pi_2(K)$, appears somewhat unexpectedly as follows. The quotient group

$$H_2(K; \mathbb{Z}) / I(\pi_2(K)),$$

where $I\colon \pi_2(K) \to H_2(K; \mathbb{Z})$ is the natural map, is always isomorphic to an abelian group, G^*, obtained from the fundamental group $G = \pi_1(K)$ in a purely algebraic manner; G^* is what is now known as the second *group* homology of the

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group G. With AP theory, in the more restricted but very important class of closed, orientable 3-manifolds, with Theorem 2, we actually obtain $\pi_2\left(M^3\right)$ *completely* in its totality, just purely algebraically, from a certain type of presentation, an Artin presentation, of the fundamental group $\pi_1\left(M^3\right)$. The same remarks also apply to Corollary 5 of [10] and Theorem A2.1 of [11]. Namely, we concentrate only on closed, orientable 3-manifolds, instead of more general finite simplicial complexes. However, we obtain clearer and sharper results due to the fundamental purely group-theoretic Artin equation which, moreover, is related to the deep 4D smooth topological Donaldson Theorem.

2. Theorem 2 substitutes for the still unresolved Wall's Conjecture [1, p. 19] in the sense that it characterizes, purely algebraically, the fundamental groups of closed, orientable, *irreducible* 3-manifolds.
3. Theorem 1 begs the question: *why should $SU(2)$ be the only Lie group surviving in the discreteness of AP theory?*

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