Feedback

On 106.06: Robert M. Young and Jack Calcut write: It is always a delight to stumble serendipitously on an optimisation problem without a solution. As a companion to the results in this Note, we offer the following surprising and counter-intuitive theorem which stands in stark contrast to the classic calculus problem involving one or two cones.

Theorem: Let the unit disk be partitioned into three sectors, each of which is used to form a right circular cone. There is no partition for which the combined volume of the cones is a maximum.

Solution: Toward that end, fix a partition, call the angles of the sectors a, b, c, and let $V(a)$, $V(b)$, $V(c)$ denote the volumes of the corresponding cones. Note that

$$(a + b) + (b + c) + (c + a) = 4\pi$$

so that at least one term on the left cannot exceed $\frac{4}{3}\pi$; suppose that

$$a + b \leqslant \frac{4\pi}{3}.$$

It is sufficient to establish the following inequality:

$$V(a) + V(b) < V(a + b). \tag{1}$$

In other words, by combining the two smallest sectors into one, a cone of greater volume is always obtained. Conclusion:

$$V(a) + V(b) + V(c) < V(a + b) + V(c)$$

and hence by continuity we can choose ε so small that

$$V(a) + V(b) + V(c) < V(a + b) + V(c - \varepsilon) + V(\varepsilon).$$

This shows that no partition can produce a maximal volume.

To establish (1), we shall make use of the following simple lemma, which does not appear to be well known.

Lemma: If f is strictly increasing on the open interval (A, B), where A is NONnegative, then the function $g(x) = xf(x)$ is *strictly superadditive* on that interval, namely,

$$g(x) + g(y) < g(x + y)$$

whenever x, y, $x + y$ belong to (A, B).

The proof is immediate:

$$\begin{aligned} g(x) + g(y) &= xf(x) + yf(y) \\ &< xf(x + y) + yf(x + y) \\ &= (x + y)f(x + y) = g(x + y) \end{aligned}$$

since x and y are positive.

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Since the volume V of a cone of radius x and slant height 1 is given by

$$V = \tfrac{1}{3}\pi x^2\sqrt{1 - x^2},$$

it remains only to show that the function $g(x) = x^2\sqrt{1-x^2}$ is strictly superadditive on a sufficiently large interval $(0, B)$. Write $g(x) = xf(x)$, where

$$f(x) = x\sqrt{1 - x^2}$$
$$= \sqrt{x^2(1 - x^2)}.$$

Since the function $\sqrt{u(1 - u)}$ is strictly increasing on $0 < u < \tfrac{1}{2}$, it follows that g is strictly superadditive on $0 < x < \sqrt{\tfrac{1}{2}}$.

Finally, let r and s denote the radii of the cones corresponding to the two sectors with angles a and b, respectively. Then

$$a = 2\pi r \qquad \text{and} \qquad b = 2\pi s.$$

Accordingly,

$$2\pi(r + s) = a + b \leqslant \tfrac{4}{3}\pi$$

so that

$$r + s \leqslant \tfrac{2}{3}.$$

The proof is over:

$$\frac{2}{3} < \sqrt{\frac{1}{2}}$$

and (1) follows at once.

Remarks

A simple induction argument shows that the n sector problem never has an optimal solution when $n > 2$. Moreover, in this case the combined volume of the n corresponding cones is always less than the maximal volume obtainable with two cones. As is well known, the optimal solution for the single cone problem need not rely on calculus: it follows almost at once from the inequality between the arithmetic and geometric means. It would be interesting to know if the two cone problem also has a purely algebraic solution.

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On 106.17: Alan Beardon writes: In Note 106.17 the authors seek all polynomials p that satisfy the functional relation

$$p(x)^2 = 1 + p(x + 1)p(x - 1).$$

First, by elementary manipulation, they reduce this to the relation $p(x)q(x + 1) = p(x + 1)q(x)$, where $q(x) = p(x + 1) + p(x - 1)$. Their argument then proceeds by several steps, each of which is based on finding,

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