Feedback

On 106.06: Robert M. Young and Jack Calcut write: It is always a delight to stumble serendipitously on an optimisation problem without a solution. As a companion to the results in this Note, we offer the following surprising and counter-intuitive theorem which stands in stark contrast to the classic calculus problem involving one or two cones.

**Theorem:** Let the unit disk be partitioned into three sectors, each of which is used to form a right circular cone. There is no partition for which the combined volume of the cones is a maximum.

**Solution:** Toward that end, fix a partition, call the angles of the sectors , , and let , , denote the volumes of the corresponding cones. Note that

\[ (a + b) + (b + c) + (c + a) = 4\pi \]

so that at least one term on the left cannot exceed \( \frac{4\pi}{3} \); suppose that

\[ a + b \leq \frac{4\pi}{3}. \]

It is sufficient to establish the following inequality:

\[ V(a) + V(b) < V(a + b). \] (1)

In other words, by combining the two smallest sectors into one, a cone of greater volume is always obtained. Conclusion:

\[ V(a) + V(b) + V(c) < V(a + b) + V(c) \]

and hence by continuity we can choose \( \epsilon \) so small that

\[ V(a) + V(b) + V(c) < V(a + b) + V(c - \epsilon) + V(\epsilon). \]

This shows that no partition can produce a maximal volume.

To establish (1), we shall make use of the following simple lemma, which does not appear to be well known.

**Lemma:** If \( f \) is strictly increasing on the open interval \((A, B)\), where \( A \) is negative, then the function \( g(x) = xf(x) \) is strictly superadditive on that interval, namely,

\[ g(x) + g(y) < g(x + y) \]

whenever \( x, y, x + y \) belong to \((A, B)\).

The proof is immediate:

\[ g(x) + g(y) = xf(x) + yf(y) \]
\[ < xf(x + y) + yf(x + y) \]
\[ = (x + y)f(x + y) = g(x + y) \]

since \( x \) and \( y \) are positive.

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Since the volume \( V \) of a cone of radius \( x \) and slant height 1 is given by
\[
V = \frac{1}{3}\pi x^2 \sqrt{1 - x^2},
\]
it remains only to show that the function \( g(x) = x^2 \sqrt{1 - x^2} \) is strictly superadditive on a sufficiently large interval \((0, B)\). Write \( g(x) = x f(x) \), where
\[
f(x) = x \sqrt{1 - x^2} = \sqrt{x^2 (1 - x^2)}.
\]
Since the function \( \sqrt{u (1 - u)} \) is strictly increasing on \( 0 < u < \frac{1}{2} \), it follows that \( g \) is strictly superadditive on \( 0 < x < \sqrt{\frac{1}{2}} \).

Finally, let \( r \) and \( s \) denote the radii of the cones corresponding to the two sectors with angles \( a \) and \( b \), respectively. Then
\[
a = 2\pi r \quad \text{and} \quad b = 2\pi s.
\]
Accordingly,
\[
2\pi (r + s) = a + b \leq \frac{4}{3}\pi
\]
so that
\[
r + s \leq \frac{2}{3}.
\]
The proof is over:
\[
\frac{2}{3} < \frac{1}{\sqrt{2}}
\]
and (1) follows at once.

Remarks

A simple induction argument shows that the \( n \) sector problem never has an optimal solution when \( n > 2 \). Moreover, in this case the combined volume of the \( n \) corresponding cones is always less than the maximal volume obtainable with two cones. As is well known, the optimal solution for the single cone problem need not rely on calculus: it follows almost at once from the inequality between the arithmetic and geometric means. It would be interesting to know if the two cone problem also has a purely algebraic solution.

On 106.17: Alan Beardon writes: In Note 106.17 the authors seek all polynomials \( p \) that satisfy the functional relation
\[
p(x)^2 = 1 + p(x + 1)p(x - 1).
\]
First, by elementary manipulation, they reduce this to the relation
\[
p(x)q(x + 1) = p(x + 1)q(x), \quad \text{where} \quad q(x) = p(x + 1) + p(x - 1).
\]
Their argument then proceeds by several steps, each of which is based on finding,